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An Introduction to the Mechanics of Fluids

C. Truesdell
K.R. Rajagopal

Modeling and Simulation in
Science, Engineering and Technology

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C. Truesdell
K.R. Rajagopal

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[While] the creative power of pure thought is at work, the outside world asserts itself again; through the real phenomena it forces new questions upon us; it opens up new fields of mathematical science; and while we try to gain these new fields of science for the realm of pure thought, we often find the answers to old unsolved problems and so at the time best further the old theories. . . .

Besides it is wrong to think that rigor in proof is the enemy of simplicity. Numerous examples establish the opposite, that the rigorous method is also the simpler and the easier to grasp. The pursuit of rigor compels us to discover simpler arguments; also, often it clears the path to methods susceptible of more development than were the old, less rigorous ones. . . .

While I insist upon rigor in proofs as a requirement for a perfect solution of a problem, I should like, on the other hand, to oppose the opinion that only the concepts of analysis, or even those of arithmetic alone, are susceptible of a fully rigorous treatment. This opinion, occasionally advocated by eminent men, I consider entirely mistaken. Such a one-sided interpretation of the requirement of rigor would soon lead us to ignore all concepts that derive from geometry, mechanics, and physics, to shut off the flow of new material from the outside world, and finally, indeed, as a last consequence, to reject the concepts of the continuum and of the irrational number. What an important, vital nerve would be cut, were we to root out geometry and mathematical physics! On the contrary, I think that wherever mathematical ideas come up, whether from the theory of knowledge or in geometry, or from the theories of natural science, the task is set for mathematics to investigate the principles underlying these ideas and establish them upon a simple and complete system of axioms in such a way that in exactness and in application to proof the new ideas shall be no whit inferior to the old arithmetical concepts.

To new concepts correspond, necessarily, new symbols. Those we choose in such a way that they remind us of the phenomena which gave rise to the formation of the new concepts. . . .

If we do not succeed in solving a mathematical problem, it is often because we have failed to recognize the more general standpoint from which the problem before us appears only as a single link in a chain of related problems. . . . This way to find general methods is certainly the most practicable and the surest, for he who seeks for methods without having a definite problem in mind mainly seeks in vain.

A role still more important than generalization's in dealing with mathematical problems is played, I believe, by specialization. Perhaps in most cases where we seek in vain for the answer to a question the cause of failure lies in our having not yet or not completely solved problems simpler and easier than the one in hand. Everything depends then on finding these easier problems and solving them by use of tools as perfect as possible and of concepts susceptible to generalization. This rule is one of the most important levers for overcoming mathematical difficulties . . . .

[The] conviction that every mathematical problem can be solved is a powerful incentive to us as we work. We hear within us the perpetual call: *There is the problem. Seek its solution. You can find it by pure thinking, for in mathematics there is no ignorabimus!*

Hilbert
Mathematical Problems
Archiv für Mathematik und Physik (3) 1,
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Preface

In their origins, hydrodynamics and acoustics were sciences largely mathematical. In them during the eighteenth century originated, and for them was developed, much of the theories of partial-differential equations and the kinematics of continuous media. From them in the early nineteenth century grew most of the theory of elasticity and, later, electrostatics and electrodynamics. Until the end of the period closed by the First World War, every mathematician and every physicist inclined to theory mastered their elements as a matter of course, and most research journals in mathematics or physics published research enlarging them. For outlines of what a beginner in physics was expected to learn about continuum mechanics, some ninety or more years ago, we may look at Part III of Webster's *Dynamics*, 1904, and Joos's *Theoretical Physics*, 1932 correct, simple, clear, immortal.

As applications and experimental studies grew more numerous, and as publication rather than mastery became essential to nutrience of multitudes of employees, specialists proliferated, their abcdarians were trained more and more in their specialties alone, and the old science of continuum mechanics was silted over by an alluvium of verbose, intricate ramifications, each said to be a profession. After the Second World War, "applied mathematicians" arose to provide in ever increasing, baffling abundance precise, rigorous theorems of existence, uniqueness and failure of it, regularity, stability and instability. To comprehend the very statements they announce, advanced knowledge of modern analysis is required. Often these difficult analytical researches, which employ a setting in one or another function space claimed natural to the problem studied and solved, are products of some institute.

In this book we have endeavored to provide a compact and moderately general foundation of the mechanics of continua, turning aside now and then to particular applications that rise to hand as illustrations of some general principles. Here, we proffer some applications special to fluids, first of rather general kinds and then for the classical fluids named after Euler, Navier, and Stokes. Certainly we do not denigrate approximate theories and numerical work, but since they dominate most recent books on hydrodynamics, aerodynamics, and acoustics, we have chosen to set before the student a bit of mathematically exact work, if only to let him see that some of it formerly studied is still good, and that some more recent progress in that old-fashioned way may yet enlighten and serve.

No mathematical analysis beyond that commonly taught to undergraduates who have learn mathematics in mathematics departments is needed.

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1

Bodies, Configurations, and Motions

1.1 Introduction[1]

A body $\mathcal{B}$ is a set that has a topological structure and a measure structure. It is assumed to be a σ-finite measure space with a nonnegative measure $\mu(\mathcal{P})$ defined over a σ-ring of subsets $\mathcal{P}$ of $\mathcal{B}$ called subparts of the body. The open sets of $\mathcal{B}$ are assumed to be the σ-ring of sets. The members of the smallest σ-ring containing the open sets are called Borel sets of $\mathcal{B}$.

A one-to-one mapping $\chi : \mathcal{B} \times I \to \mathcal{E}$, where $I \in (-\infty, t_o)$ for some t_o, and $\mathcal{E}$ denotes three-dimensional Euclidean space, or more explicitly

$$\mathbf{x} = \chi(X, t), \tag{1.1-1}$$

is called a *motion* of $\mathcal{B}$. Here X is a particle, t is the time ($-\infty < t < \infty$), and $\mathbf{x}$ is a place in Euclidean space. The value of χ is the place $\mathbf{x}$ that the particle X comes to occupy at the time t. The notations $\mathcal{B}_\chi$ and $\mathcal{P}_\chi$ will indicate the configurations of $\mathcal{B}$ and $\mathcal{P}$ at the time t. We shall consider only motions that are smooth in the sense that χ is differentiable with respect to X and t as many times as may be needed. The *velocity* $\dot{\mathbf{x}}$, *acceleration* $\ddot{\mathbf{x}}$, and nth *acceleration* $\overset{(n)}{\mathbf{x}}$ of the particle X at the time t are defined as usual—

$$\begin{aligned} \dot{\mathbf{x}} &:= \partial_t \chi(X, t), \\ \ddot{\mathbf{x}} &:= \partial_t^2 \chi(X, t), \\ \overset{(n)}{\mathbf{x}} &:= \partial_t^n \chi(X, t), \end{aligned} \tag{1.1-2}$$

[1]The concepts of bodies, configurations and motions are developed in great detail in chapter I, C. Truesdell, "*A First Course in Rational Continuum Mechanics*," vol. 1 (New York: Academic Press, 1991).

—so that

$$\dot{\mathbf{x}} = \overset{(1)}{\mathbf{x}}, \ddot{\mathbf{x}} = \overset{(2)}{\mathbf{x}}.$$

By the Radon-Nikodym theorem, the mass of $\mathcal{P}$ may be expressed in terms of a *mass density* ρ_χ,

$$M(\mathcal{P}) = \int_{\mathcal{P}_\chi} \rho_\chi \, \mathrm{dv}, \tag{1.1-3}$$

where the integral is in the sense of the Lebesgue integral. Clearly the function ρ_χ depends on χ, and, as indicated, the integration is carried out over the configuration $\mathcal{P}_\chi$ of $\mathcal{P}$.

The existence of a mass density expresses a relation between the abstract body $\mathcal{B}$ and the configuration $\mathcal{B}_\chi$ it occupies. For a suitably chosen sequence of parts $\mathcal{P}_k$, nested so that $\mathcal{P}_{k+1} \subset \mathcal{P}_k$, that all the $\mathcal{P}_k$ have but the single point $\mathbf{x}$ in common, and that the volume $V(\mathcal{P}_k)$ approaches 0 as $k \to \infty$, the density is the ultimate ratio of mass to volume:

$$\rho_\chi(\mathbf{x}, t) = \lim_{k\to\infty} \frac{M(\mathcal{P}_k)}{V(\mathcal{P}_k)}. \tag{1.1-4}$$

To find the relation between the mass densities corresponding to different configurations of the same part $\mathcal{P}$, we begin with the formulae

$$M(\mathcal{P}) = \int_{\mathcal{P}_{\chi_1}} \rho_{\chi_1} \mathrm{dv} = \int_{\mathcal{P}_{\chi_2}} \rho_{\chi_2} \mathrm{dv}. \tag{1.1-5}$$

If we let $\boldsymbol{\lambda}$ stand for the mapping that carries $\chi_1(\mathcal{B})$ into $\chi_2(\mathcal{B})$ and write J for the absolute value of its Jacobian determinant,

$$J \equiv |\det \nabla \boldsymbol{\lambda}|, \tag{1.1-6}$$

then

$$\int_{\mathcal{P}_{\chi_1}} \rho_{\chi_1} \mathrm{dv} = \int_{\mathcal{P}_{\chi_1}} \rho_{\chi_2} J \mathrm{dv}, \tag{1.1-7}$$

for every part $\mathcal{P}$. Thence follows an equation relating the two densities:

$$\rho_{\chi_1} J = \rho_{\chi_2}. \tag{1.1-8}$$

This equation shows that the density in any one configuration determines the densities in all others.

1.2 Reference Configuration

Often it is convenient to select one particular configuration and refer everything concerning the body to that configuration, which need be only a possible one, not one ever occupied by the body. Let κ be a mapping of the abstract body $\mathcal{B}$ into three-dimensional Euclidean space, namely, a placement. Then the mapping

$$\mathbf{X} = \kappa(X) \tag{1.2-1}$$

gives the place $\mathbf{X}$ occupied by the particle X in the configuration $\chi(\mathcal{B})$. Since the mapping is smooth, by assumption,

$$X = \kappa^{-1}(\mathbf{X}). \tag{1.2-2}$$

Hence the motion (1.1-1) may be written in the form

$$\mathbf{x} = \chi[\kappa^{-1}(\mathbf{X}), t] \equiv \chi_\kappa(\mathbf{X}, t). \tag{1.2-3}$$

In the description furnished by this equation, the motion is a sequence of mappings of the *reference configuration* $\kappa(\mathcal{B})$ onto the actual configuration $\chi(\mathcal{B})$. Thus the motion is visualized as mapping parts of space onto parts of space. A reference configuration is introduced to allow us to employ the apparatus of Euclidean geometry.

The choice of reference configuration, like the choice of a coordinate system, is arbitrary. The reference configuration, which may be any smooth image of the body, need not even be a configuration ever occupied by the body in the course of its motion. For each different κ, a different function χ_κ results in (3). Thus *one* motion of the body is represented by *infinitely many* different motions of parts of space, one for each choice of κ. For some choice of κ, we may get a particularly simple description, just as in geometry one choice of coordinates may lead to a simple equation for a particular figure, but the reference configuration itself has nothing to do with such motions as it may be used to describe, just as the coordinate system has nothing to do with geometrical figures themselves. A reference configuration is introduced to allow the use of mathematical apparatus familiar in other contexts. Again there is an analogy to coordinate geometry, where coordinates are introduced, not because they are natural or germane to geometry but because they allow the familiar apparatus of algebra to be applied at once.

1.3 Descriptions of Motion

There are four methods of describing the motion of a body: the material, the referential, the spatial, and the relative (discussed in Section 1.7). Because of our hypotheses of smoothness, all are equivalent.

In the *material description* we deal directly with the abstract particles X. This description corresponds to the only one used in analytical dynamics, where we always speak of the first, second, ..., nth masses. To be precise, we should say, "the mass point X_i whose mass is m_i," but commonly this expression is abbreviated to "the mass i" or "the body m_i". In a continuous body $\mathcal{B}$ there are infinitely many particles X. In the material description, the independent variables are X and t, the particle and the time. While the material description is the most natural in concept, it was used in continuum mechanics until 1951 but is not used much now. For some time the term "material description" was used to denote another and older description often confused with it, the description to which we turn next.

The *referential description* employs a reference configuration. In the mid-eighteenth century Euler introduced the description that hydrodynamicists still call "Lagrangean." This is a particular referential description in which the Cartesian coordinates of the position $\mathbf{X}$ of the particle X at the time $t = 0$ are used as a label for the particle X. It was recognized that such labeling by initial coordinates is arbitrary, and writers on the foundations of hydrodynamics have often mentioned that the results must be and are independent of the choice of the initial time, and some have remarked that the parameters of any triple system of surfaces moving with the material will do just as well. The referential description, taking $\mathbf{X}$ and t as independent variables, includes all these possibilities. The referential description, in some form, is always used in classical elasticity theory, and the best studies of the foundations of classical hydrodynamics have employed it almost without fail. It is the description commonly used in modern works on continuum mechanics, and we shall use it in this book. We must always bear in mind that the choice of $\kappa(\mathcal{B})$ is ours, that $\kappa(\mathcal{B})$ is merely some configuration that the body might occupy, and that physically significant results must be independent of the choice of $\kappa(\mathcal{B})$. Any motion has infinitely many different referential descriptions, equally valid.

In the *spatial description*, attention is focused on the present configuration of the body, the region of space currently occupied by the body. This description, which was introduced by d'Alembert, is called "Eulerian" by the hydrodynamicists. The place $\mathbf{x}$ and the time t are taken as independent variables. Since (1.1-1) is invertible,

$$X = \chi^{-1}(\mathbf{x}, t), \tag{1.3-1}$$

any function $f(X, t)$ may be replaced by a function of $\mathbf{x}$ and t:

$$f(X, t) = f[\chi^{-1}(\mathbf{x}, t), t] \equiv F(\mathbf{x}, t). \tag{1.3-2}$$

The function F, moreover, is unique. Thus, while there are infinitely many referential descriptions of a given motion, there is only one spatial description. With the spatial description, we watch what is occurring in a fixed region of space as time goes on. This description seems perfectly suited to studies of fluids, where often a rapidly deforming mass comes from no one knows where and goes no one knows whither, so that we may prefer to consider what is going on here and now. However convenient kinematically, the spatial description is awkward for questions of principle in mechanics, since in fact what is happening to the body, not to the region of space occupied by the body, enters the laws of dynamics. This difficulty is reflected by the mathematical gymnastics writers of textbooks on aerodynamics often go through in order to get formulae that are easy and obvious in the material or referential descriptions.

According to (2), the value of any smooth function of the particles of $\mathcal{B}$ at time t is given also by a field defined over the configuration $\mathcal{B}_\chi$ at time t. In this way, for example, we obtain from (1.1-2) the *velocity field* and the *acceleration field*:

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, t), \quad \ddot{\mathbf{x}} = \ddot{\mathbf{x}}(\mathbf{x}, t). \tag{1.3-3}$$

Here we have written $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ in two senses each: as the field functions and also as their values.

1.4 Deformation Gradient

The gradient of χ_κ in (1.2-3) is called the *deformation gradient* $\mathbf{F}$:

$$\mathbf{F} := \mathbf{F}_\kappa(\mathbf{X}, t) := \nabla\chi_\kappa(\mathbf{X}, t). \tag{1.4-1}$$

It is the linear approximation of the mapping χ_κ. More precisely, we should call it the gradient of the deformation from $\kappa(\mathcal{B})$ to $\chi(\mathcal{B})$, but when, as is usual, a single reference configuration $\kappa(\mathcal{B})$ is laid down and kept fixed, no confusion should result from the failure to remind ourselves that the very concept of a deformation gradient presumes use of a reference configuration.

Going back to (1.1-8), we derive EULER's *referential equation for the density*,

$$\rho J = \rho_\kappa, \tag{1.4-2}$$

where ρ is written for ρ_χ, the mass density in the present configuration, and where

$$J := |\det \mathbf{F}|. \tag{1.4-3}$$

Henceforth J will be used in the sense just defined rather than in the more general one expressed by (1.1-6). Since χ_κ is invertible, $\det \mathbf{F}$ is of one sign for all $\mathbf{X}$ and t, for a given reference configuration $\kappa(\mathcal{B})$. While (2) is often called "the Lagrangean equation of continuity," that name is misleading, since if the motion *is* spatially smooth, (2) holds, but if the motion is *not* spatially smooth, generally J cannot be defined at all, so (2) becomes impossible to consider as a condition. The proper way to interpret (2) is to regard it as a condition giving the present density ρ, once the density ρ_κ in the reference configuration is known.

EXERCISE 1.4.1

By using the formula for differentiating a determinant, derive the following identity of Euler:

$$\dot{J} = J \operatorname{div} \dot{\mathbf{x}}, \tag{1.4-4}$$

where the dot denotes the time-derivative of $J(\mathbf{X}, t)$ and where $\operatorname{div} \dot{\mathbf{x}}$ is the divergence of the velocity field $(1.3\text{-}3)_1$.

If we differentiate (2) with respect to time, regarding all quantities in it as functions of $\mathbf{X}$ and t, and then use (4), we obtain d'Alembert and Euler's *spatial equation for the density*:

$$\dot{\rho} + \rho \operatorname{div} \dot{\mathbf{x}} = 0. \tag{1.4-5}$$

This equation has exactly the same meaning as (2), which, conversely, may be obtained from it by integration.

EXERCISE 1.4.2

A motion is called *isochoric* if the volume $V(\mathcal{P})$ of each part $\mathcal{P}$ of the body remains constant in time. Show that any one of the following three equations is a necessary and sufficient condition for isochoric motion:

$$\operatorname{div} \dot{\mathbf{x}} = 0, \rho = \rho_\kappa, J = 1. \tag{1.4-6}$$

In plane flow the velocity is everywhere parallel to a given plane and is the same at all points on each line normal to the plane. To study plane flow, it suffices to confine attention to the fields of velocity and acceleration restricted to any one place.

EXERCISE 1.4.3 (D'Alembert, Noll)

If the boundary of a region on which plane flow $\dot{\mathbf{x}}$ is defined is the union of a finite number of curves in rigid motion, and if (for an infinite region) there is no flux into or out of the region of the plane beyond some sufficiently large circle, show that the general solution of $(6)_1$ is given in terms of a stream function q by

$$\dot{\mathbf{x}} = (\nabla q)^{\perp}, \tag{1.4-7}$$

where ∇ denotes the gradient operator in the plane and $\perp$ denotes rotating counterclockwise through a right angle about the normal to the plane. The velocity $\dot{\mathbf{x}}$ is tangent to the curve $q(\cdot, t) = \text{const.}$ through each $\mathbf{x}$ and t.

1.5 Material Time Rates and Gradients in the Spatial Description

In continuum mechanics the need to distinguish a vast number of quantities often deprives us of the luxury of using for a function a symbol different from that for its value, as logically we ought to do. If two functions of different variables have the same value and if *both* are denoted by that value, when we come to differentiate it is not clear which function is intended. The distinction, which of course is essential, is made by introducing different symbols for the differential operators. Henceforth $\dot{f}$ and ∇f will be used to denote the partial time derivative and the gradient of the function $\tilde{f}(\mathbf{X}, t)$ such that

$$f = \tilde{f}(\mathbf{X}, t), \tag{1.5-1}$$

while $\partial_t f$ and grad f shall denote the time derivative and the gradient of the function $\hat{f}(\mathbf{x}, t)$ that has the same value as $\tilde{f}$, namely,

$$f = \hat{f}(\mathbf{x}, t). \tag{1.5-2}$$

Since $\mathbf{x} = \chi_\kappa(\mathbf{X}, t)$, application of the chain rule to the equation $\tilde{f}(\mathbf{X}, t) = \hat{f}(\mathbf{x}, t)$ yields the classical formula of Euler:

$$\dot{f} = \partial_t f + (\text{grad } f)\dot{\mathbf{x}}. \tag{1.5-3}$$

In particular, the acceleration $\ddot{\mathbf{x}}$ is calculated from the velocity field $\dot{\mathbf{x}}(\mathbf{x}, t)$ by the formula

$$\ddot{\mathbf{x}} = \partial_t \dot{\mathbf{x}} + (\text{grad } \dot{\mathbf{x}})\dot{\mathbf{x}}. \tag{1.5-4}$$

EXERCISE 1.5.1
If f is a scalar-valued function, then show that

$$\nabla f = \mathbf{F}^T \text{ grad } f. \tag{1.5-5}$$

The notations "div" and "curl" will be used only in the spatial description, and superimposed (n) shall stand for n superimposed dots. The notation "Div" shall stand for the divergence formed from ∇ and the notation "div" for the divergence from grad.

It follows from (3) that (1.4-5) can be expressed in the forms

$$\rho' + (\text{grad } \rho) \cdot \dot{\mathbf{x}} + \rho \text{ div } \dot{\mathbf{x}} = 0, \quad \rho' + \text{div}(\rho \dot{\mathbf{x}}) = 0, \tag{1.5-6}$$

where $(\cdot)'$ denotes partial derivative with respect to time with $\mathbf{x}$ held fixed.

1.6 Change of Reference Configuration

Let the same motion (1.1-1) be described in terms of two different reference configurations, $\kappa_1(\mathcal{B})$ and $\kappa_2(\mathcal{B})$:

$$\mathbf{x} = \chi_{\kappa_1}(\mathbf{X}, t) = \chi_{\kappa_2}(\mathbf{X}, t). \tag{1.6-1}$$

The deformation gradients $\mathbf{F}_1$ and $\mathbf{F}_2$ at $\mathbf{X}$, t are of course generally different. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ denote the positions of X in $\kappa_1(\mathcal{B})$ and $\kappa_2(\mathcal{B})$:

$$\mathbf{X}_1 = \kappa_1(X), \quad \mathbf{X}_2 = \kappa_2(X). \tag{1.6-2}$$

Then

$$\mathbf{X}_2 = \kappa_2[\kappa_1^{-1}(\mathbf{X}_1)] := \boldsymbol{\lambda}(\mathbf{X}_1), \tag{1.6-3}$$

say. The deformation from κ_1 to χ can be effected in two ways: either straight off by use of χ_{κ_1}, or by using $\boldsymbol{\lambda}$ to get to κ_2 and then using χ_{κ_2} to get to χ. If $\circ$ denotes the composition of mappings, then

$$\chi_{\kappa_1} = \chi_{\kappa_2} \circ \boldsymbol{\lambda}. \tag{1.6-4}$$

Since this relation holds among the three mappings, their linear approximations, the gradients, are related through

$$\mathbf{F}_1 = \mathbf{F}_2\mathbf{P}, \tag{1.6-5}$$

where

$$\mathbf{P} \equiv \nabla\boldsymbol{\lambda}. \tag{1.6-6}$$

Of course, the relation (5) expresses the "chain rule" of differential calculus.

1.7 Current Configuration as Reference

To serve as a reference, a configuration need only be a diffeomorph of the body. So far, we have employed a reference configuration fixed in time, but we could just as well use a varying one. Thus one motion may be described in terms of any other. The only variable reference configuration really useful in this way is the present one. If we take it as a reference, we describe the past and future as they seem to an observer fixed to the particle X now at the place $\mathbf{x}$. The corresponding description is called *relative*.

To see how such a description is constructed, consider the configurations of $\mathcal{B}$ at the two times t and τ:

$$\boldsymbol{\xi} = \chi(X, \tau),$$
$$\mathbf{x} = \chi(X, t). \tag{1.7-1}$$

That is, $\boldsymbol{\xi}$ is the place occupied at time τ by the particle that at time t occupies $\mathbf{x}$:

$$\boldsymbol{\xi} = \chi[\chi^{-1}(\mathbf{x}, t), \tau],$$
$$:= \chi_t(\mathbf{x}, \tau), \tag{1.7-2}$$

say. The function χ_t just defined is called the *relative deformation function.*

Sometimes we shall wish to calculate the relative deformation function when the motion is given to us only through the spatial description of the velocity field:

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, t). \tag{1.7-3}$$

By $(1)_1$,

$$\partial_\tau \boldsymbol{\xi} = \dot{\mathbf{x}}(\boldsymbol{\xi}, \tau). \tag{1.7-4}$$

Since the right-hand side is a given function, we thus have a differential equation to integrate. The initial condition to be satisfied by the integral $\boldsymbol{\xi} = \chi_t(\mathbf{x}, t)$ is

$$\boldsymbol{\xi}|_{\tau=t} = \chi_t(\mathbf{x}, t) = \mathbf{x}. \tag{1.7-5}$$

When the motion is described by (2), we shall use a subscript t to denote quantities derived from χ_t. Thus $\mathbf{F}_t$, defined by

$$\mathbf{F}_t := \mathbf{F}_t(\tau) := \operatorname{grad} \chi_t, \tag{1.7-6}$$

is the *relative deformation gradient.*

By (1.6-5), at $\mathbf{X}$,

$$\mathbf{F}(\tau) = \mathbf{F}_t(\tau)\mathbf{F}(t). \tag{1.7-7}$$

As the fixed reference configuration with respect to which $\mathbf{F}(t)$ and $\mathbf{F}(\tau)$ are taken, we may select the configuration occupied by the body at time t'. Then (7) yields

$$\mathbf{F}_{t'}(\tau) = \mathbf{F}_t(\tau)\mathbf{F}_{t'}(t). \tag{1.7-8}$$

Of course,

$$\mathbf{F}_t(t) = \mathbf{1}. \tag{1.7-9}$$

1.8 Boundaries and Boundary Conditions

A stationary surface $\mathcal{Y}_\kappa$ in the reference shape $\kappa(\mathcal{B})$ is described by an equation of the form $f(\mathbf{X}) = 0$, and hence

$$\dot{f} = 0. \tag{1.8-1}$$

Conversely, if (1) is satisfied by a function $f(\mathbf{X}, t)$, then in fact the surface $f = 0$ is a stationary surface in the reference shape, provided of course that $\mathbf{X} \in \kappa(\mathcal{B})$. At the time t, the substantial points that make up $\mathcal{Y}_\kappa$ constitute a certain surface $\mathcal{Y}$ in the shape assumed by $\mathcal{B}$ in its motion at the time t. These surfaces are the successive forms of a single *substantial surface*. In accord with the convention we have established, we write f also for the function of $\mathbf{x}$ and t whose value at $\chi_\kappa(\mathbf{X}, t)$ is $f(\mathbf{X})$, so for the locus $f = 0$ to represent a substantial surface, we have the necessary and sufficient condition (1), where now the operation signified by a dot is defined by (1.5-3). Thus in the spatial description this requirement becomes Euler's condition:

$$f' + (\operatorname{grad} f) \cdot \dot{\mathbf{x}} = 0. \tag{1.8-2}$$

If $\mathbf{n}$ is the oriented unit normal to the surface $f = 0$, where of course f now stands for the function such that $f(\mathbf{x}, t) = 0$ is the locus of $\mathcal{Y}$, then (2) may be written alternatively in the form

$$S_n = \mathbf{n} \cdot \dot{\mathbf{x}}, \tag{1.8-3}$$

provided S_n, which is called the *speed of displacement* of $\mathcal{Y}$, is the speed at which that surface advances in the direction normal to itself in space:

$$S_n = \frac{-f'}{|\operatorname{grad} f|}. \tag{1.8-4}$$

Euler's condition (2) thus asserts that the speed of displacement of $\mathcal{Y}$ at $(\mathbf{x}, t)$ is just the same as the speed at which the substantial point now occupying $(\mathbf{x}, t)$ is moving in the direction normal to $\mathcal{Y}$.

EXERCISE 1.8.1

Let a surface $\mathcal{Y}$ have parametric representation $\mathbf{x} = \mathbf{g}(\mathbf{A}, t)$, the parameter $\mathbf{A}$ being an ordered pair of real parameters. If $\mathbf{A}$ is regarded as permanently denoting a particular point on $\mathcal{Y}$ as $\mathcal{Y}$ moves, shows that its velocity $\mathbf{u}$ satisfies $\mathbf{n} \cdot \mathbf{u} = S_n$. If $\mathcal{Y}$ is represented by some spatial equation, say $h(\mathbf{x}, t) = 0$, the same field S_n is obtained in this way. This fact justifies the name "speed of displacement."

EXERCISE 1.8.2 (Lagrange)
If (2) is regarded as a partial differential equation for f in the spatial description, show that integration by the method of characteristics yields $F(\chi_\kappa^{-1}, f) = 0$. Thus, the substantial points that lie upon $f(\mathbf{x}, t) = \text{const}$, at any one time, lie always upon its image under the notion.

A *kinematic boundary* is a surface that separates permanently two parts of $\mathcal{B}$. Thus a kinematic boundary is a substantial surface, and vice versa. The special term "boundary" is introduced to distinguish particular surfaces, usually assigned in advance, like a wall or a surface with a special role such as separating two parts having different properties. The simplest example is a *stationary wall*, a surface $f(\mathbf{x}) = \text{const}$. In order for such a surface to be substantial and hence a possible kinematic boundary for a given motion of $\mathcal{B}$, by (3) we have the following necessary and sufficient condition relating the unit normal $\mathbf{n}$ to the velocity:

$$\mathbf{n} \cdot \dot{\mathbf{x}} = 0. \tag{1.8-5}$$

That is, *the velocity field on the wall is tangential*, as is obvious. More generally, if the places on a wall have assigned velocities $\mathbf{u}$, then at those places

$$\mathbf{n} \cdot \dot{\mathbf{x}} = \mathbf{n} \cdot \mathbf{u}. \tag{1.8-6}$$

Sometimes a stronger kinematic condition is imposed, that of *adherence*. The body is then constrained to move with the kinematic boundary. If the places on the wall have an assigned velocity $\mathbf{v}$, then on that wall

$$\dot{\mathbf{x}} = \mathbf{v}. \tag{1.8-7}$$

In the case of a stationary wall, this condition becomes

$$\dot{\mathbf{x}} = \mathbf{0}. \tag{1.8-8}$$

EXERCISE 1.8.3
Let the surface $\mathcal{Y}$ whose equation is $g(\mathbf{x}, t) = 0$ in $\chi(\mathcal{B}, t)$ be the image of the surface $\mathcal{Y}_\kappa$ whose equation is $G(\mathbf{X}, t) = 0$ in the reference shape $\kappa(\mathcal{B})$. (Note that $\mathcal{Y}_\kappa$, in contradistinction with substantial surfaces, generally moves with respect to $\kappa(\mathcal{B})$.) With the conventions of notation set at the beginning of this section, show that the oriented unit normals $\mathbf{n}_\kappa$ and $\mathbf{n}$ to these two surfaces are related by

$$\mathbf{n}_\kappa = \frac{|\operatorname{grad} g|}{|\operatorname{Grad} g|}\mathbf{F}^T\mathbf{n}; \tag{1.8-9}$$

and that the speed of advance S_κ of the surface $\mathcal{Y}_\kappa$ in the direction normal to itself in $\kappa(\mathcal{B})$ is given by

$$S_\kappa = -\frac{\dot{g}}{|\operatorname{Grad} g|}; \tag{1.8-10}$$

and

$$S_\kappa = \frac{|\operatorname{grad} g|}{|\operatorname{Grad} g|}(S_n - \mathbf{n} \cdot \dot{\mathbf{x}}). \tag{1.8-11}$$

2

Kinematics and Basic Laws

2.1 Stretch and Rotation

Since the motion χ_κ is smooth and, **F** is nonsingular, the polar decomposition theorem of Cauchy enables us to write it in the two forms

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.1-1}$$

where **R** is an orthogonal tensor and **U** and **V** are positive-definite symmetric tensors. **R**, **U**, and **V** are unique. Cauchy's decomposition tells us that the deformation corresponding locally to **F** may be obtained by effecting pure stretches of amounts, say, v_i, along three suitable mutually orthogonal directions $\mathbf{e}_i$, followed by a rigid rotation of those directions, or by performing the same rotation first and then effecting the same stretches along the resulting directions. The quantities v_i are the *principal stretches*; corresponding unit proper vectors of **U** and **V** point along the *principal axes of strain* in the reference configuration and the present configuration χ, respectively. Indeed, if

$$\mathbf{U}\mathbf{e}_i = v_i\mathbf{e}_i, \tag{2.1-2}$$

then by (1)

$$\mathbf{V}(\mathbf{R}\mathbf{e}_i) = (\mathbf{R}\mathbf{U}\mathbf{R}^T)(\mathbf{R}\mathbf{e}_i) = v_i(\mathbf{R}\mathbf{e}_i). \tag{2.1-3}$$

Thus, as just asserted, **U** and **V** have common proper numbers but different principal axes, and **R** is the rotation that carries the principal axes of **U** into the principal axes of **V**. **R** is orthogonal but need not be proper orthogonal: $\mathbf{RR}^T = 1$, so $\det \mathbf{R} = +1$ or -1, and $\det \mathbf{R}$ maintains either the one value or the other for all **X** and t, by continuity. Thus $\det \mathbf{U} = \det \mathbf{V} = |\det \mathbf{F}| = J$.

R is called the *rotation tensor*, **U** and **V** the *right* and *left stretch tensors*, respectively. These tensors, like **F** itself, are to be interpreted as comparing aspects of the present configuration with their counterparts in the reference configuration.

The *right* and *left Cauchy-Green tensors* **C** and **B** are defined as follows:

$$\mathbf{C} := \mathbf{U}^2 = \mathbf{F}^T\mathbf{F},$$
$$\mathbf{B} := \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T. \tag{2.1-4}$$

While the fundamental decomposition (1) plays the major part in the proof of general theorems, calculation of **U**, **V**, and **R** in special cases may be awkward, since operations such as taking the square root are usually required. **C** and **B**, however, are calculated by mere multiplication of **F** and $\mathbf{F}^T$. If g_{km} and $g^{\alpha\beta}$ are the covariant and contravariant metric components in arbitrarily selected coordinate systems in space and in the reference configuration, respectively, components of **C** and **B** are

$$C_{\alpha\beta} = F^k_\alpha F^m_\beta g_{km},$$
$$B^{km} = F^k_\alpha F^m_\beta g^{\alpha\beta}, \tag{2.1-5}$$

where $F^k_\alpha = x,^k_\alpha \equiv \partial_{X^\alpha} \chi^k_\kappa(X_1, X_2, X_3, t)$. The proper numbers of **C** and **B** are the squares of the principal stretches, v_i^2. The principal invariants of **C** and **B** are given by

$$I = \operatorname{tr}\mathbf{B} = \operatorname{tr}\mathbf{C} = v_1^2 + v_2^2 + v_3^2,$$
$$II = \frac{1}{2}[(\operatorname{tr}\mathbf{B})^2 - \operatorname{tr}\mathbf{B}^2] = \frac{1}{2}[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2] = v_1^2v_2^2 + v_2^2v_3^2 + v_3^2v_1^2,$$
$$III = \det\mathbf{B} = \det\mathbf{C} = J^2 = v_1^2v_2^2v_3^2. \tag{2.1-6}$$

If we begin with the relative deformation $\mathbf{F}_t$, defined by (1.7-6), and apply to it the polar decomposition theorem, we obtain the *relative rotation* $\mathbf{R}_t$, the *relative stretch tensors* $\mathbf{U}_t$ and $\mathbf{V}_t$, and the *relative Cauchy-Green tensors* $\mathbf{C}_t$ and $\mathbf{B}_t$:

$$\mathbf{F}_t = \mathbf{R}_t\mathbf{U}_t = \mathbf{V}_t\mathbf{R}_t, \quad \mathbf{C}_t = \mathbf{U}_t^2, \qquad \mathbf{B}_t = \mathbf{V}_t^2. \tag{2.1-7}$$

We now define what is meant by the history associated with a function. The restriction of the function $f(\tau)$ to times τ not later than the present time t is called the *history* of f up to time t and is denoted by $f^t(s)$ or f^t:

$$f^t := f^t(s) := f(t-s), \ t \text{ fixed}, \ s \geq 0. \tag{2.1-8}$$

The history f^t, as its name suggests, is the portion of a function of all time that corresponds to the present and past times only. Histories turn out to be of major importance in mechanics because it is the present and past that determine the future.

In the notation (2.1-8), for example, $\mathbf{C}_t^t$ (s) is the history of the relative right Cauchy-Green tensor up to time t.

2.2 Stretching and Spin

For a tensor defined from the relative motion, for example $\mathbf{F}_t$, we introduce the notation

$$\dot{\mathbf{F}}_t(t) := \partial_\tau \mathbf{F}_t(\tau)|_{\tau=t} = -\partial_s \mathbf{F}_t^t(s)|_{s=0}. \tag{2.2-1}$$

Set

$$\begin{aligned} \mathbf{G} &:= \dot{\mathbf{F}}_t(t), \\ \mathbf{D} &:= \dot{\mathbf{U}}_t(t) = \dot{\mathbf{V}}_t(t), \\ \mathbf{W} &:= \dot{\mathbf{R}}_t(t). \end{aligned} \tag{2.2-2}$$

$\mathbf{D}$, which is called the *stretching*, is the rate of change of the stretch of the configuration at time $t + \varepsilon$ with respect to that at time t, in the limit as $\varepsilon \to 0$. Likewise, $\mathbf{W}$, which is called the *spin*, is the ultimate rate of change of the rotation from the present configuration to one occupied just before or just afterward. Since $\mathbf{U}_t$ is symmetric, so is $\mathbf{D}$, being its derivative with respect to a parameter

$$\mathbf{D}^T = \mathbf{D}, \tag{2.2-3}$$

but, unlike $\mathbf{U}_t$, $\mathbf{D}$ generally fails to be positive-definite. If we differentiate the relation $\mathbf{R}_t(\tau)\mathbf{R}_t(\tau)^T = \mathbf{1}$ with respect to τ, put $\tau = t$, and use $(2)_3$, we find that $\mathbf{W}$ is skew:

$$\mathbf{W}^T + \mathbf{W} = 0. \tag{2.2-4}$$

From its definition $(2)_1$, $\mathbf{G}$ is the ultimate rate of change of $\mathbf{F}_t$, but that is not all, for by (1.7-7) we have

$$\mathbf{G} = \dot{\mathbf{F}}(t)\mathbf{F}(t)^{-1}. \tag{2.2-5}$$

Differentiation of (1.4-1) with respect to t yields

$$\dot{\mathbf{F}} = \nabla \dot{\chi}_\kappa = \nabla \dot{\mathbf{x}} = (\text{grad } \dot{\mathbf{x}})\mathbf{F}, \tag{2.2-6}$$

where the last step follows by the chain rule. Substitution into (5) yields

$$\mathbf{G} = \text{grad } \dot{\mathbf{x}}. \tag{2.2-7}$$

We have shown that the tensor $\mathbf{G}$, defined by $(2)_1$, is in fact the spatial *velocity gradient*.

EXERCISE 2.2.1
Prove that if

$$\mathbf{G}_n := \overset{(n)}{\mathbf{F}}_t(t), \tag{2.2-8}$$

then

$$\mathbf{G}_n = \operatorname{grad} \overset{(n)}{\mathbf{x}} . \tag{2.2-9}$$

If we differentiate the polar decomposition $(2.1\text{-}7)_1$ with respect to τ and then put $\tau = t$, we find that

$$\mathbf{G} = \mathbf{D} + \mathbf{W}. \tag{2.2-10}$$

This result, showing that **D** and **W** are the symmetric and skew parts of the velocity gradient, expresses the fundamental *Euler-Stokes decomposition* of the instantaneous motion into the sum of a pure stretching along three mutually orthogonal axes and a rigid spin of those axes.

Of course, we could have defined **G** by (7) as the velocity gradient and **W** and **D** by (10) as the symmetric and skew parts of **G**. We should then have had to prove $(2)_{2,4}$ as theorems to interpret **G**, **W**, and **D** kinematically. Writers on hydrodynamics usually prefer the argument in this order.

Motions in which $\mathbf{W} = \mathbf{0}$ are called *irrotational.* They form the main subject of study in classical hydrodynamics.

Since **W** is skew, it may be represented by an axial vector, denoted by $\boldsymbol{\omega} = \operatorname{curl} \dot{\mathbf{x}}$, called the "vorticity" in hydrodynamics. Nowadays it seems more convenient not to introduce this vector but instead to use the tensor **W**.

Further enlightenment of the difference between stretch and stretching and between rotation and spin is furnished by the following exercise.

EXERCISE 2.2.2

Prove that

$$\begin{aligned} \mathbf{W} &= \dot{\mathbf{R}}\mathbf{R}^T + \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T, \\ \mathbf{D} &= \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T, \end{aligned} \tag{2.2-11}$$

where **R** and **U** have their usual meanings as the rotation and right stretch tensors with respect to a fixed reference configuration.

Clearly the spin **W** is generally something quite different from $\dot{\mathbf{R}}$, the time rate of the rotation tensor, and the stretching **D** is entirely different from $\dot{\mathbf{U}}$, the time rate of the stretch tensor. If the simple equations (11) had been available to the hydrodynamicists of the nineteenth century, a long and acrimonious controversy in the literature could have been avoided. Helmholtz interpreted **W** as an instantaneous rotation; Bertrand objected because a simple shear flow is rotational in Helmholtz's sense even though the particles move in straight lines, while if the particles rotate in concentric circles with an appropriate distribution of velocities, the motion is irrotational. Such controversy disappears by a glance at (11), which shows that the conditions $\dot{\mathbf{R}} = \mathbf{0}$ and $\mathbf{W} = \mathbf{0}$ are far from the same. The definitions (2.1-1) and (2) make the different kinematic meanings of $\mathbf{W}$ and $\dot{\mathbf{R}}$ clear and suggest that both tensors will be useful.

Higher rates of change of stretch and rotation may be defined in various ways. The most useful higher rates are the *Rivlin-Ericksen tensors* $\mathbf{A}_n$:

$$\mathbf{A}_n := \overset{(n)}{\mathbf{C}_t}(t), \tag{2.2-12}$$

where the notation (1.1-2) is used.

In a steady, simple shearing, the Cartesian components of the velocity are

$$\dot{x}_1 = 0, \dot{x}_2 = \kappa x_2, \dot{x}_3 = 0, \kappa = \text{const.}, \tag{2.2-13}$$

κ is the shearing, and each material point moves ahead at constant speed along a straight line along the x_2-axis. The motion is rotational unless $\kappa = 0$.

In a steady, simple vortex the contravariant components in cylindrical coordinates are

$$\dot{r} = 0, \dot{\theta} = \omega(r), \dot{z} = 0, \tag{2.2-14}$$

and each material point rotates steadily about the polar axis on a circle r = constant, z = constant at the angular speed $\omega(r)$. If $\omega(r) = Kr^{-2}$, the motion is irrotational, where K is the strength of the irrotational vortex. The vorticity is given by $r\omega = (r^2\omega)'$, where prime denotes derivative with respect to r.

Let

$$\mathbf{W}_a := \text{skw grad } \ddot{\mathbf{x}}. \tag{2.2-15}$$

EXERCISE 2.2.3 (D'Alembert, Euler, Beltrami)
Show that

$$\mathbf{W}_a := \dot{\mathbf{W}} + \mathbf{DW} + \mathbf{WD}. \tag{2.2-16}$$

EXERCISE 2.2.4 (Dupont, Rivlin, and Ericksen)
Show that

$$\mathbf{A}_n = \mathbf{G}_n + \mathbf{G}_n^T + \sum_{j=1}^{n-1} \binom{n}{j} \mathbf{G}_j^T \mathbf{G}_{n-j}; \tag{2.2-17}$$

and

$$\mathbf{A}_{n+1} = \dot{\mathbf{A}}_n + \mathbf{A}_n \mathbf{G} + \mathbf{G}^T \mathbf{A}_n. \tag{2.2-18}$$

We now consider restrictions imposed upon $\mathbf{W}$ by boundaries. To do so, we first appeal to Kelvin's transformation ("Stokes's theorem")[1] of an integral over a surface $\mathcal{Y}$ into a line integral around the border of $\mathcal{Y}$, which we shall denote by $\mathcal{C}$. The *circulation* $C(\mathcal{C})$ was introduced by Kelvin as a measure of the summed tangential speeds of the substantial points lying presently upon $\mathcal{C}$. Assuming that $\dim \mathcal{E} = 3$, we suppose the surface $\mathcal{Y}$ to be given by a mapping $\mathbf{x} = \mathbf{f}(a, b)$ on a domain $\mathcal{D}$ of the parameters a and b. Then, with the usual convention of sign and on the assumption that the fields and the surface be sufficiently smooth,

$$C(\mathcal{C}) := \int_{\mathcal{C}} \dot{\mathbf{x}} \cdot d\mathbf{x} = \int_{\mathcal{D}} \mathbf{W} \cdot (\partial_a \mathbf{x} \wedge \partial_b \mathbf{x})\, da\, db. \tag{2.2-19}$$

For our first use of this statement, we apply it to a surface $\mathcal{Y}$ that is normal to the velocity field $\dot{\mathbf{x}}$. Then $C(\mathcal{C}) = 0$, so the right-hand side of (17) vanishes. The same holds for every subsurface of $\mathcal{Y}$. If $\mathbf{W}$ and $\partial_a \mathbf{x} \wedge \partial_b \mathbf{x}$ are continuous, then everywhere on $\mathcal{Y}$

$$\mathbf{W} \cdot (\partial_a \mathbf{x} \wedge \partial_b \mathbf{x}) = 0. \tag{2.2-20}$$

We have proved the following theorem: *At a point on a surface normal to the velocity field, either* $\mathbf{W} = \mathbf{0}$ *or the axis of* $\mathbf{W}$ *lies in the tangent plane.*

EXERCISE 2.2.5

If $\mathbf{n}$ is a unit normal field to the surface, show that

$$\mathbf{W} = -\mathbf{n} \otimes \mathbf{W}\mathbf{n} + \mathbf{W}\mathbf{n} \otimes \mathbf{n}. \tag{2.2-21}$$

The foregoing statement holds *a fortiori* on a stationary boundary to which a body adheres.

EXERCISE 2.2.6 (Weatherburn, Berker, Caswell, Truesdell)

Interpretation of the gradient in terms of the directional derivative shows that if $\mathbf{k}$ is any vector in the tangent plane at the place $\mathbf{x}$ on a stationary wall to which a body adheres, then at $\mathbf{x}$

$$\mathbf{G}\mathbf{k} = \mathbf{0}.$$

Hence show that at $\mathbf{x}$

$$\mathbf{D} = E\mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{W}\mathbf{n} + \mathbf{W}\mathbf{n} \otimes \mathbf{n}, \tag{2.2-22}$$

---

[1] A surface is a compact, oriented, two-dimensional manifold with boundary in a three-dimensional Euclidean space. The velocity field $\dot{\mathbf{x}}$ is assumed to be differentiable in an open set properly containing $\mathcal{Y}$. A brief statement and a rigorous proof of Kelvin's transformation are given by M. Spivak, *Calculus on Manifolds* (Benjamin: New York, 1965).

and the principal stretchings are given by

$$\begin{aligned} 2D_1 &= E + \sqrt{E^2 + \omega^2} \geq 0, \\ D_2 &= D_{(e)} = 0, \\ 2D_3 &= E - \sqrt{E^2 + \omega^2} \leq 0, \end{aligned} \tag{2.2-22}$$

where $\omega = |\operatorname{curl} \dot{\mathbf{x}}|$.

EXERCISE 2.2.7 (Cauchy)
If

$$\mathbf{W}_a := \operatorname{skw} \mathbf{G}_2 = \operatorname{skw} \operatorname{grad} \ddot{\mathbf{x}}, \tag{2.2-23}$$

then show that

$$(\mathbf{F}^T \mathbf{W} \mathbf{F})^{\cdot} = \mathbf{F}^T \mathbf{W}_a \mathbf{F}. \tag{2.2-24}$$

Hence a necessary and sufficient condition that $\mathbf{F}^T\mathbf{W}\mathbf{F}$ remain constant for each substantial point X in the course of its motion is

$$\mathbf{W}_a = \mathbf{0}. \tag{2.2-25}$$

If (25) holds, then

$$\mathbf{F}^T \mathbf{W} \mathbf{F} = \mathbf{f}, \tag{2.2-26}$$

a function of place $\mathbf{X}$ in the reference shape. Because $\mathbf{F} = \mathbf{1}$ throughout that shape, from (26), we conclude that $\mathbf{f} = \mathbf{W}_\kappa$, the spin that X would have, were it to be at $\mathbf{X}$. In particular, (25) is satisfied by an irrotational flow.

The condition (25) is of central importance in classical fluid dynamics. There it is applied in a region, not merely to a single substantial point. It is called the *D'Alembert-Euler condition.* A convenient way to express it is

$$\operatorname{skw} \operatorname{grad} \ddot{\mathbf{x}} = \mathbf{0}; \tag{2.2-27}$$

because of (9), equivalently

$$\operatorname{skw} \ddot{\mathbf{F}}\mathbf{F}^{-1} = \operatorname{skw}(\dot{\mathbf{G}} + \mathbf{G}^2) = \mathbf{0}. \tag{2.2-28}$$

For the time being we remark only that according to a familiar theorem on lamellar fields, in a simply connected region the field $\ddot{\mathbf{x}}$ satisfies (25) if and only if there is an *acceleration potential* $P_{\mathbf{a}}$:

$$\ddot{\mathbf{x}} = -\operatorname{grad} P_{\mathbf{a}}. \tag{2.2-29}$$

EXERCISE 2.2.8 (D'Alembert, Euler, Beltrami)
Show that

$$\mathbf{W}_a = \dot{\mathbf{W}} + \mathbf{D}\mathbf{W} + \mathbf{W}\mathbf{D}. \tag{2.2-30}$$

EXERCISE 2.2.9 (Appell)
If $\dim \mathcal{E} = 3$,

$$\left(\frac{1}{2}|J\mathbf{W}|^2\right)^{\bullet} = J^2(\mathbf{W}\cdot\mathbf{W}_{\mathrm{a}} + |\mathbf{W}|^2\mathbf{n}\cdot\mathbf{D}\mathbf{n}), \tag{2.2-31}$$

where $\mathbf{n}$ is either unit vector in the nullspace of $\mathbf{W}$. Hence $w = \|\operatorname{curl}\ \dot{\mathbf{x}}\|$ satisfies the differential equation

$$(J\omega)^{\bullet} = J\omega\mathbf{n}\cdot\mathbf{D}\mathbf{n} \tag{2.2-32}$$

if and only if

$$\mathbf{W}\cdot\mathbf{W}_a = 0. \tag{2.2-33}$$

EXERCISE 2.2.10
Show that a rigid motion has an acceleration potential if and only if its spin is steady, and then

$$-P_{\mathrm{a}} = \frac{1}{2}\mathbf{p}\cdot\mathbf{W}^2\mathbf{p} + [\dot{\mathbf{c}} + \mathbf{W}(\mathbf{c} - \dot{\mathbf{x}}_0)]\cdot\mathbf{p}; \tag{2.2-34}$$

here $\mathbf{p} := \mathbf{x} - \mathbf{x}_0$. If ω denotes the angular speed and r the distance from the axis of spin,

$$\mathbf{p}\cdot\mathbf{W}^2\mathbf{p} = \frac{1}{4}w^2r^2 = -\omega^2r^2. \tag{2.2-35}$$

As we have seen earlier in this section, the condition $\mathbf{W} = \mathbf{0}$ defines an irrotational motion. Consequently, a motion is irrotational in a simply connected region if and only if it has there a *velocity potential* P_{v}:

$$\dot{\mathbf{x}} = -\operatorname{grad} P_{\mathrm{v}}. \tag{2.2-36}$$

For that reason irrotational motions are often called *potential flows*. The potential P_{v} may depend upon t as well as $\mathbf{x}$. The surfaces $P_{\mathrm{v}}(t, \mathbf{x}) = \text{const.}$, t fixed, are called *equipotentials*. The velocity is normal to the equipotential on which it lies. A system of equipotentials determined by giving to P_{v} successively equal, constant increments, say c, divides the region of flow into laminae, and hence an irrotational flow is sometimes called *lamellar*. If the constant c is very small, so also are the values of the function d that delivers the normal distances between the equipotentials, and $|\dot{\mathbf{x}}| \approx c/d$.

If an irrotational motion is also isochoric, then, as Euler remarked, $(1.4\text{-}6)_1$ reduces to the linear partial differential equation later to be called "Laplace's equation":

$$\Delta P_{\mathrm{v}} = 0. \tag{2.2-37}$$

Solutions, which are called *harmonic functions*, are easy to obtain. The sum of two harmonic functions is a harmonic function, so the outcome of superposing two isochoric, irrotational flows is likewise an isochoric, irrotational flow, and complicated flows may be built up from simple ones in this way. In the nineteenth century many general properties of them were discovered, and general methods for calculating solutions of (37) such as to satisfy $\mathbf{n} \cdot \dot{\mathbf{x}} = 0$ on given boundaries were constructed. The corpus of these properties is called "potential theory." The problem of determining an isochoric, irrotational flow within or about assigned boundaries is purely kinematical; it can be phrased with no reference to mechanics.

A disquieting property of isochoric, irrotational flows is revealed by a theorem in the theory of the "Laplacian" equation: The boundary condition $\mathbf{n} \cdot \dot{\mathbf{x}} = 0$ applied to the boundary of a closed, bounded, simply connected region determines a unique velocity field in that region. Were the fluid to adhere to some bounding wall we should have to prescribe $\dot{\mathbf{x}}$, not merely $\mathbf{n} \cdot \dot{\mathbf{x}}$. A standard theorem of potential theory may be interpreted as follows: *If at a certain time a body undergoing isochoric, irrotational flow adheres to a not void, open set on a surface, that whole body must be at rest at that time.*

Neither isochoric motion nor the condition of adherence nor the restriction to a bounded domain is necessary to render impossible an irrotational motion other than a state of rest, as is shown by the following, purely kinematical theorem.

Theorem of Kelvin and Helmholtz. *Let an irrotational flow in a stationary, simply connected region be such that*

1. *It is isochoric, or its density is steady.*

2. *On all finite boundaries,* $\dot{\mathbf{x}} \cdot \mathbf{n} = 0$.

3. *In any part of the region that lies outside of a sphere of arbitrarily large radius r, if the motion is isochoric, then*

$$P_{\mathbf{v}} \partial_r P_{\mathbf{v}} = o\left(\frac{1}{r^2}\right) \quad \text{as } r \to \infty, \tag{2.2-38}$$

while if the density is steady,

$$\rho P_{\mathbf{v}} \partial_r P_{\mathbf{v}} = o\left(\frac{1}{r^2}\right) \quad \text{as } r \to \infty. \tag{2.2-39}$$

Then $\dot{\mathbf{x}} = \mathbf{0}$ *everywhere.*

Of course the two main conditions, those of isochoric motion and of steady density, are not mutually exclusive, for it is easily possible that both (38) and (39) hold.[2]

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[2]For a proof of this theorem see C. Truesdell *A First Course in Rational Continuum Mechanics*, vol. 1, (New York: Academic Press, 1991).

A measure of the relative strengths of the rotation and stretching is given by the *vorticity number* $\mathcal{W}$ that is defined through

$$\mathcal{W} = \frac{\|\mathbf{W}\|}{\|\mathbf{D}\|}. \tag{2.2-40}$$

If $\mathbf{D} = \mathbf{0}$ and $\mathbf{W} \neq 0$, we choose to say that $\mathcal{W} = \infty$.

2.3 Changes of Frame

In classical mechanics we think of an observer as being a rigid body carrying a clock. Actually we need not an observer as such but only the concept of change of observer, or, as we shall say, change of frame. The ordered pair $\{\mathbf{x}, t\}$, a place and a time, is called an *event*. The totality of events is *space-time*. A *change of frame* is a one-to-one mapping of space-time onto itself such that distances, time intervals, and the sense of time are preserved. We expect that every such transformation should be a time-dependent orthogonal transformation of space combined with a shift of the origin of time. This is so. The most general change of frame is given by

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_0),$$
$$t^* = t - a, \tag{2.3-1}$$

where $\mathbf{c}(t)$ is a time-dependent point, $\mathbf{Q}(t)$ is a time-dependent orthogonal tensor, $\mathbf{x}_0$ is a fixed point, and a is a constant. We commonly say that $\mathbf{c}(t)$ represents a change of origin (translation), since the fixed point $\mathbf{x}_0$ is mapped into $\mathbf{c}(t)$. $\mathbf{Q}(t)$ represents a rotation and also, possibly, a reflection. Reflections are included since, although in most physics and engineering courses the student is taught to use a right-handed coordinate system, there is nothing in nature to prevent two observers from orienting themselves oppositely.

A frame need not be defined and certainly must not be confused with a coordinate system. It is convenient, however, to describe (1) as a change from "the unstarred frame to the starred frame," since this wording promotes the interpretation in terms of two different observers.

A quantity is said to be *frame indifferent* if it meets the following requirements:

$$\mathrm{A}^* = \mathrm{A} \text{ for indifferent scalars,}$$
$$\mathbf{v}^* = \mathbf{Q}\mathbf{v} \text{ for indifferent vectors,}$$
$$\mathbf{S}^* = \mathbf{Q}\mathbf{S}\mathbf{Q}^T \text{ for indifferent tensors (of second order).}$$

An indifferent scalar is a quantity that does not change its value. An indifferent vector is one that is the same "arrow" in the sense that

$$\text{if } \mathbf{v} = \mathbf{x} - \mathbf{y}, \text{ then } \mathbf{v}^* = \mathbf{x}^* - \mathbf{y}^*. \tag{2.3-2}$$

By (1), then,

$$\mathbf{v}^* = \mathbf{Q}(\mathbf{x} - \mathbf{y}) = \mathbf{Q}\mathbf{v}, \tag{2.3-3}$$

as asserted. An indifferent tensor is one that transforms indifferent vectors into indifferent vectors. That is,

$$\text{if } \mathbf{v} = \mathbf{S}\mathbf{w} \text{ and } \mathbf{v}^* = \mathbf{Q}\mathbf{v},\ \mathbf{w}^* = \mathbf{Q}\mathbf{w},\ \text{then } \mathbf{v}^* = \mathbf{S}^*\mathbf{w}^*. \tag{2.3-4}$$

By substituting the first three equations into the last, we find that

$$\mathbf{Q}\mathbf{v} = \mathbf{S}^*\mathbf{Q}\mathbf{w} = \mathbf{Q}\mathbf{S}\mathbf{w}. \tag{2.3-5}$$

Since this relation is to hold for all $\mathbf{w}$, we infer the rule $\mathbf{S}^*\mathbf{Q} = \mathbf{Q}\mathbf{S}$, as stated.

In mechanics, we meet some quantities that are indifferent and some that are not. Sometimes we have a vector or tensor defined in one frame only. By using the stated rules, we may extend the definition to all frames to obtain a frame-indifferent quantity. Such an extension is trivial. Usually, however, we are given a definition valid in all frames from the start. In that case, we have to find out what transformation law is obeyed and thus determine whether or not the quantity is frame indifferent.

Consider, for example, the motion of a body. Under the change of frame (1), (1.1-1) becomes

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)[\chi(\mathbf{X}, t) - \mathbf{x}_0] \equiv \chi^*(\mathbf{X}, t^*). \tag{2.3-6}$$

Differentiation with respect to t^* yields

$$\dot{\mathbf{x}}^* = \dot{\mathbf{c}} + \dot{\mathbf{Q}}[\mathbf{x} - \mathbf{x}_0] + \mathbf{Q}\dot{\mathbf{x}}, \tag{2.3-7}$$

so

$$\dot{\mathbf{x}}^* - \mathbf{Q}\dot{\mathbf{x}} = \dot{\mathbf{c}} + \mathbf{A}(\mathbf{x}^* - \mathbf{c}), \tag{2.3-8}$$

where

$$\mathbf{A} = \dot{\mathbf{Q}}\mathbf{Q}^T = -\mathbf{A}^T. \tag{2.3-9}$$

$\mathbf{A}$ is called the *angular velocity* or *spin* of the starred frame with respect to the unstarred one. The result (8) shows that the velocity is not a frame indifferent quantity, since if it were, the right-hand side of (8) would have to be replaced by $\mathbf{0}$. Likewise, the acceleration is not frame indifferent.

EXERCISE 2.3.1

Prove that

$$\ddot{\mathbf{x}}^* - \mathbf{Q}\ddot{\mathbf{x}} = \ddot{\mathbf{c}} + 2\mathbf{A}(\dot{\mathbf{x}}^* - \dot{\mathbf{c}}) + (\mathbf{A} + \mathbf{A}^2)(\mathbf{x}^* - \mathbf{c}). \tag{2.3-10}$$

In (6) we may refer both the motions χ and $\chi*$ to the same reference configuration κ if we wish to. This amounts to replacing χ and $\chi*$ by χ_κ and χ_κ^*, respectively, since these functions have the same values as the former. By taking the gradient of the resulting formula, we obtain

$$\mathbf{F}^* = \mathbf{QF}. \tag{2.3-11}$$

From the polar decomposition (2.1-1), then,

$$\mathbf{R}^*\mathbf{U}^* = \mathbf{QRU}. \tag{2.3-12}$$

Since **QR** is orthogonal and since a polar decomposition is unique,

$$\mathbf{R}^* = \mathbf{QR}, \quad \text{and } \mathbf{U}^* = \mathbf{U}. \tag{2.3-13}$$

Hence

$$\mathbf{V}^* = \mathbf{R}^*\mathbf{U}^*\mathbf{R}^{*^T} = \mathbf{QRU}(\mathbf{QR})^T, = \mathbf{QVQ}^T. \tag{2.3-14}$$

Thus we have shown that **V** is frame indifferent, while **F**, **R**, and **U** are not. It follows from $(13)_2$, (14) and the definitions (2.1-4) that $\mathbf{C}^* = \mathbf{C}$ and $\mathbf{B}^* = \mathbf{QBQ}^T$. If we differentiate (11) with respect to time, we find that

$$\dot{\mathbf{F}}^* = \mathbf{Q}\dot{\mathbf{F}} + \dot{\mathbf{Q}}\mathbf{F}, \tag{2.3-15}$$

but by (2.2-5), $\dot{\mathbf{F}} = \mathbf{GF}$, so

$$\mathbf{G}^*\mathbf{F}^* = \mathbf{QGF} + \dot{\mathbf{Q}}\mathbf{F} = \mathbf{QGQ}^T\mathbf{F}^* + \dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{F}^*. \tag{2.3-16}$$

Since $\mathbf{F}^*$ is non singular, it may be canceled from this equation, which by use of (2.2.10) becomes

$$\mathbf{D}^* + \mathbf{W}^* = \mathbf{Q}(\mathbf{D} + \mathbf{W})\mathbf{Q}^T + \mathbf{A}, \tag{2.3-17}$$

where **A** is the spin (9) of the starred frame with respect to the unstarred one. Since a decomposition into symmetric and skew parts is unique,

$$\mathbf{D}^* = \mathbf{QDQ}^T, \mathbf{W}^* = \mathbf{QWQ}^T + \mathbf{A}. \tag{2.3-18}$$

These formulae embody the *theorem of Zaremba*: The stretching is frame indifferent, while the spin in the starred frame is the sum of the spin in the unstarred frame and the spin of the starred frame with respect to the unstarred frame. The assertion is intuitively plain, since a change of frame is a rigid motion, which does not alter the stretchings of elements though it does rotate the directions in which they seem to occur.

EXERCISE 2.3.2

Prove that the Rivlin-Ericksen tensors $\mathbf{A}_n$ are frame indifferent.

Most relations of interest to us shall be invariant under change of frame. To have to demand invariance of a relation under a group of transformations is a sign that we have not formulated a good mathematical language for the problem at hand, since in such a language invariance is automatic. The use of a frame in classical mechanics is artificial in that the frame and its motion really have nothing to do with the phenomenon being observed. An abstract formulation of space-time, kinematics, and dynamics, in which frames play no part, has been given by Noll. Here, we shall continue to use a formulation of mechanics close to the one taught to freshmen.

2.4 Forces and Moments

Forces and torques are among the primitive elements of mechanics; like bodies and motions, they are given *a priori*. While several kinds of forces and torques are considered in a more general mechanics, in these lectures we shall need only the classical ones.[3] Forces act on the parts $\mathcal{P}$ of a body $\mathcal{B}$ in a configuration $\chi(\mathcal{B})$. We shall require two kinds of forces: *body force* $\mathbf{f}_b(\mathcal{P})$, which is an absolutely continuous function of the volume of $\mathcal{P}_\chi$, and *contact force* $\mathbf{f}_c(\mathcal{P})$, which is an absolutely continuous function of the surface area of the boundary $\partial\mathcal{P}_\chi$ of $\mathcal{P}_\chi$. The *resultant force* $\mathbf{f}(\mathcal{P})$ acting on $\mathcal{P}$ in χ is given by

$$\mathbf{f}(\mathcal{P}) = \mathbf{f}_b(\mathcal{P}) + \mathbf{f}_c(\mathcal{P}), \tag{2.4-1}$$

where

$$\begin{aligned} \mathbf{f}_b(\mathcal{P}) &= \int_{\mathcal{P}_\chi} \mathbf{b}\mathrm{dm} = \int_{\mathcal{P}_\chi} \rho\mathbf{b}\mathrm{dv}, \\ \mathbf{f}_c(\mathcal{P}) &= \int_{\partial\mathcal{P}_\chi} \mathbf{t}\mathrm{ds}. \end{aligned} \tag{2.4-2}$$

The two densities $\mathbf{b}$ and $\mathbf{t}$ are called the *specific body force* and the *traction*, respectively.

The *resultant moment of force* $\mathbf{L}(\mathcal{P}; \mathbf{x}_0)$ with respect to $\mathbf{x}_0$ is defined by

$$\mathbf{L}(\mathcal{P}, \mathbf{x}_0) = \int_{\mathcal{P}_\chi} (\mathbf{x} - \mathbf{x}_0) \wedge \mathbf{b}\mathrm{dm} + \int_{\partial\mathcal{P}_\chi} (\mathbf{x} - \mathbf{x}_0) \wedge \mathbf{t}\mathrm{ds}. \tag{2.4-3}$$

A moment or *simple torque* is only a special case of a *torque*, but more general torques are not needed in an introductory course.

[3]See C. Truesdell, *A First Course in Rational Mechanics.*, vol. 1, chapter 1, section 8 (Boston: Academic Press, 1991).

EXERCISE 2.4.1
Prove that

$$\mathbf{L}(\mathcal{P}, \mathbf{x}_0') = \mathbf{L}(\mathcal{P}, \mathbf{x}_0) + (\mathbf{x}_0 - \mathbf{x}_0') \wedge \mathbf{f}(\mathcal{P}). \tag{2.4-4}$$

We assume that *forces and torques are frame indifferent.* That is, under a change of frame (2.3-1),

$$\mathbf{b}^* = \mathbf{Q}\mathbf{b} \qquad \text{and} \qquad \mathbf{t}^* = \mathbf{Q}\mathbf{t}. \tag{2.4-5}$$

2.5 Euler's Laws of Mechanics

The *momentum* $\mathbf{m}(\mathcal{P})$ and the *moment of momentum* $\mathbf{M}(\mathcal{P}; \mathbf{x}_0)$ of $\mathcal{P}$ in the configuration $\chi(\mathcal{B}, t)$ are defined by

$$\begin{aligned} \mathbf{m}(\mathcal{P}) &= \int_{\mathcal{P}_\chi} \dot{\mathbf{x}} dm, \\ \mathbf{M}(\mathcal{P}; \mathbf{x}_0) &= \int_{\mathcal{P}_\chi} (\mathbf{x} - \mathbf{x}_0) \wedge \dot{\mathbf{x}} dm. \end{aligned} \tag{2.5-1}$$

As axioms relating the forces applied to the motion produced, we lay down *Euler's laws*:

$$\begin{aligned} \mathbf{f}(\mathcal{P}) &= \dot{\mathbf{m}}(\mathcal{P}), \\ \mathbf{L}(\mathcal{P}; \mathbf{x}_0) &= \dot{\mathbf{M}}(\mathcal{P}; \mathbf{x}_0). \end{aligned} \tag{2.5-2}$$

(Of course, Euler's laws generalize and include much earlier ones, due in varying forms and circumstances to medieval schoolmen, Huygens, Newton, James Bernoulli, and others. There is no justice in emphasizing our debt to any one of these predecessors of Euler to the exclusion of the rest.) Euler's laws assert that the resultant force equals the rate of change of momentum and the resultant moment of force equals the rate of change of moment of momentum. Here $\mathcal{P}$ is a fixed part of the body $\mathcal{B}$, and $\mathbf{x}_0$ is a fixed point in space. The dots on the right-hand side of (2) indicate the ordinary time derivatives. (Of course $\mathcal{P}_\chi$ is not the fixed region in space, since it is the configuration of $\mathcal{P}$ at time t in the motion χ.)

EXERCISE 2.5.1
If $(2)_1$ holds, then show that $(2)_2$ holds for one $\mathbf{x}_0$ if and only if it holds for all $\mathbf{x}_0$.

It is possible to derive Euler's laws as a theorem from a single axiom proposed by Noll: the rate of working is frame indifferent for every part $\mathcal{P}$ of every body $\mathcal{B}$.

In modern mechanics the definitions of **m** and **M** are sometimes generalized, and torques more general than (2.4-3) are often considered. Euler's laws, or an equivalent statement, remain the fundamental axioms for all kinds of mechanics. In particular, if the sign $\int$ is taken to mean a Lebesgue-Stieltjes integral, Euler's laws may be shown to imply as corollaries the so-called Newtonian equations of analytical dynamics, provided certain assumptions are made about the forces.

Since forces and torques are frame indifferent but the rates of change of momentum and moment of mometum are not, the truth of Euler's laws as stated here is restricted to a particular class of frames. Frames of this class are called *inertial.* In physics the frame of the "fixed stars" is regarded as an inertial one.

2.6 Euler-Cauchy Stress Principle

As defined, the densities **b** and **t** in (4-2) may be extremely general:

$$\mathbf{b} = \mathbf{b}(\mathbf{x}, t, \mathcal{P}, \mathcal{B}), \qquad \mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathcal{P}, \mathcal{B}). \tag{2.6-1}$$

We shall restrict attention to body force densities that are unaffected by the presence or absence of bodies in space:

$$\mathbf{b} = \mathbf{b}(\mathbf{x}, t), \tag{2.6-2}$$

whatever $\mathcal{P}$ and $\mathcal{B}$ are. Such body forces are called *external.* (A particular kind of more general body force called *mutual,* such as universal gravitation, is sometimes included in mechanics but will not be treated here.) The particular case when $\mathbf{b}$ = constant pertains to heavy bodies. If **b** is the gradient of a scalar field, the body force is said to be *conservative*, and

$$\mathbf{b} = -\operatorname{grad} \varpi; \tag{2.6-3}$$

the scalar function ϖ is the potential for **b**. If ϖ is a potential for **b**, then so is $\varpi + h(t)$, where h is only a function of time.

We shall restrict attention also to a particular kind of contact force, namely, one such that the traction **t** at given place and time has a common value for all parts P having a common tangent plane and lying upon the same side of it:

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n}), \tag{2.6-4}$$

where **n** is the outer normal to ∂P in the configuration χ. Such tractions are called *simple.* The pioneers of continuum mechanics laboriously created the assumption (4) through abstraction from special cases. To them it seemed ultimate in generality. The assumption embodied in (2.4-1), (2.4-2), and (4) is the *Euler-Cauchy stress principle*, which is the cornerstone of classical continuum mechanics. We shall not need to depart from it here. Noll has proved that (4) is more general than it might seem. He has shown that the principle of linear momentum under weak assumptions requires the traction to be simple.

The stress principle is put to use through Cauchy's *fundamental lemma*: There is a tensor $\mathbf{T}(\mathbf{x},t)$, called the *stress tensor*, such that

$$\mathbf{t}(\mathbf{x}, t, \mathbf{n}) = \mathbf{T}(\mathbf{x}, t)\mathbf{n}. \tag{2.6-5}$$

That is, the traction $\mathbf{t}$, which at the outset was allowed to depend arbitrarily on the normal $\mathbf{n}$, is in fact a linear homogeneous function of it.

EXERCISE 2.6.1

Show that Cauchy's fundamental lemma follows from (4) and either $(2.5\text{-}2)_1$ or $(2.5\text{-}2)_2$.

2.7 Cauchy's Laws of Continuum Mechanics

As a result of Cauchy's fundamental lemma, for continua subject to simple tractions, simple torques, and external body forces we may write Euler's laws (2.5-2) in the following special and signficant forms:

$$\left(\int_{\mathcal{P}} \dot{\mathbf{x}}\mathrm{dm}\right)^{\bullet} = \int_{\partial\mathcal{P}_\chi} \mathbf{Tn}\mathrm{ds} + \int_{\mathcal{P}_\chi} \mathbf{b}\mathrm{dm}, \tag{2.7-1}$$

$$\left[\int_{\mathcal{P}} (\mathbf{x}-\mathbf{x}_0)\wedge \dot{\mathbf{x}}\mathrm{dm}\right]^{\bullet} = \int_{\partial\mathcal{P}_\chi} (\mathbf{x}-\mathbf{x}_0)\wedge \mathbf{Tn}\mathrm{ds} + \int_{\mathcal{P}_\chi} (\mathbf{x}-\mathbf{x}_0)\wedge \mathbf{b}\mathrm{dm}, \tag{2.7-2}$$

for every part $\mathcal{P}$ of every body $\mathcal{B}$. The subscript χ is omitted from $\mathcal{P}$ on the left-hand side as a reminder that the time derivative is calculated for a fixed part $\mathcal{P}$ of the body, not for a fixed region of space.

These equations are both of the form called an *equation of balance* for a tensor field ψ:

$$\left(\int_{\mathcal{P}} \psi\mathrm{dm}\right)^{\bullet} = \int_{\partial\mathcal{P}_\chi} \mathbf{E}_\psi\mathbf{n}\mathrm{ds} + \int_{\mathcal{P}_\chi} s_\psi\mathrm{dm}. \tag{2.7-3}$$

$\mathbf{E}_\psi$, a tensor of order greater by 1 than ψ, is called an *efflux* of ψ, while s_ψ is called a *source* of ψ. An equation of balance expresses the rate of growth of $\int_{\mathcal{P}} \psi\mathrm{dm}$ as the sum of two parts: a rate of flow $-\mathbf{E}_\psi$ inward through the boundary $\partial\mathcal{P}_\chi$ of the configuration $\mathcal{P}_\chi$ of $\mathcal{P}$ and a creation s_ψ in the interior of that configuration. Equations of this form occur frequently in mathematical physics. Under the assumptions of smoothness made at the beginning of this book, an equation of balance is always equivalent to a differential equation.

EXERCISE 2.7.1
Prove that

$$\left(\int_{\mathcal{P}} \psi \mathrm{dm}\right)^{\bullet} = \int_{\mathcal{P}_\chi} \dot{\psi} \mathrm{dm}, \tag{2.7-4}$$

where $\dot{\psi}$ is the derivative of ψ in the material description and hence may be calculated from (1.5-3). Show that the equation of balance (2) holds for every part $\mathcal{P}$ of every body $\mathcal{B}$ if and only if at each interior point of $\mathcal{B}$

$$\rho\dot{\psi} = \operatorname{div}\mathbf{E}_\psi + \rho s_\psi . \tag{2.7-5}$$

The general differential equation (4) is a consequence of the divergence theorem, an easy one. Textbooks for engineers and physicists still seem to prefer proofs of each special application by means of diagrams showing boxes decorated by many small arrows. Such proofs are regarded as "intuitive," apparently because they include, over and over again, a bad proof of the divergence theorem itself along with the particular application of it. "Intuition" of this kind must be bought at the price of more time and boredom than a course on modern mechanics can afford. We must remark that (4) holds subject to specific assumptions. Use of the divergence theorem requires assumptions about the region and about the fields. The assumptions about the region are satisfied here because only interior points are considered; in fact, it would suffice to consider only those parts $\mathcal{P}$ whose configurations are spheres about the point in question. The fields, however, must be smooth. It suffices to assume that $\mathbf{E}_\psi$ is continuously differentiable and that ρ, $\dot{\psi}$, and s_ψ are continuous in a sufficiently small sphere about the interior point considered. In general, (4) does not hold at points of $\partial\mathcal{B}_\chi$ or at interior points where any of the fields ρ, $\dot{\psi}$, $\mathbf{E}$, s_ψ fail to exist or are not sufficiently smooth.

If we apply (4) to $(1)_1$, we conclude at once that Euler's first law holds if and only if

$$\rho\ddot{\mathbf{x}} = \operatorname{div}\mathbf{T} + \rho\mathbf{b} \tag{2.7-6}$$

at interior points of B_χ. This equation expresses *Cauchy's first law of continuum mechanics*.

Like Euler's laws, Cauchy's first law as stated here holds only in inertial frames. To obtain a frame-indifferent statement of it, we need only replace $\ddot{\mathbf{x}}$ by $\mathbf{a}$, the frame-indifferent vector that in some inertial frame reduces to $\ddot{\mathbf{x}}$.

To apply (4) to $(1)_2$, we let $\mathbf{M}$ stand for the tensor such that $(\mathbf{x}-\mathbf{x}_0)\wedge(\mathbf{Tn}) = \mathbf{Mn}$ and conclude that Euler's second law holds if and only if

$$\begin{aligned}\rho(\mathbf{x}-\mathbf{x}_0)\wedge\ddot{\mathbf{x}} &= \operatorname{div}\mathbf{M} + \rho(\mathbf{x}-\mathbf{x}_0)\wedge\mathbf{b}\\ &= \mathbf{T}^T - \mathbf{T} + (\mathbf{x}-\mathbf{x}_0)\wedge \operatorname{div}\mathbf{T} + \rho(\mathbf{x}-\mathbf{x}_0)\wedge\mathbf{b},\end{aligned} \tag{2.7-7}$$

where the second form follows by an easy identity. If (5) holds, then

$$\mathbf{T}^T = \mathbf{T}. \tag{2.7-8}$$

The stress tensor is symmetric and vice versa. This equation expresses *Cauchy's second law of continuum mechanics.*

Since this second law, in particular, has been questioned from time to time, we pause to emphasize its special character. We began with the general laws of Euler, but we applied them only subject to some special assumptions:

1. All torques are moments of forces.
2. The traction is simple.

We assumed also that the body force is external, but that restriction is not important here. Under these assumptions, *Euler's laws are equivalent to Cauchy's laws* when χ, $\mathbf{b}$, $\rho\ddot{\mathbf{x}}$, and $\mathbf{T}$ are sufficiently smooth. Cauchy's first law expresses locally the balance of linear momentum. Cauchy's second law, if the first is satisfied, expresses locally the balance of moment of momentum.

2.8 Equivalent Processes

We now consider motions and forces that obey the laws of mechanics. Formally, the motion of a body and the forces acting upon the corresponding configurations of the body constitute a *dynamical process* if Cauchy's laws (2.7-5) and (2.7-7) are satisfied. If the body and its mass distribution are given, Cauchy's first law (2.7-5) determines a unique body force $\mathbf{b}$. Here we wish to consider rather the totality of all possible problems. There is then no reason to restrict $\mathbf{b}$. Consequently, any pair of functions $\{\chi, \mathbf{T}\}$, where χ is a mapping of the body $\mathcal{B}$ onto configurations in space and where $\mathbf{T}$ is any smooth field of symmetric tensors defined at each time t over the configuration $\mathcal{B}_\chi$, defines a dynamical process.

Under a change of frame, χ becomes χ^* as given by (2.3-6). Our assumption (2.4-5) asserts that body force and contact force are indifferent. Since, of course, the normal $\mathbf{n}$ is indifferent, so that by (2.3-5), $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$. Two dynamical processes $\{\chi, \mathbf{T}\}$ and $\{\chi^*, \mathbf{T}^*\}$ related in this way are regarded as the same motion and associated contact forces as seen by two different observers. Thus, formally, two dynamical processes $\{\chi, \mathbf{T}\}$ and $\{\chi^*, \mathbf{T}^*\}$ will be called *equivalent* if they are related as follows:

$$\chi^*(X, t^*) = \mathbf{c}(t) + \mathbf{Q}(t)[\chi(X, t) - \mathbf{x}_0],$$

$$t^* = \mathrm{t} - \mathrm{a} \tag{2.8-1}$$

$$\mathbf{T}^*(X, t^*) = \mathbf{Q}(t)\mathbf{T}(X, t)\mathbf{Q}(t)^T. \tag{2.8-2}$$

We here regard the stress tensor as a function of the particle X and the time t.

3

Constitutive Equations, Reduced Constitutive Equations, and Internal Constraints

3.1 Constitutive Equations

The general principles of mechanics apply to all bodies and motions, and the diversity of materials in nature is represented in the theory by *constitutive equations*. A constitutive equation is a relation between forces and motions. In popular terms, forces applied to a body "cause" it to undergo a motion, and the motion "caused" differs according to the nature of the body. In continuum mechanics the forces of interest are contact forces, which are specified by the stress tensor **T**. Just as different figures are defined in geometry as idealizations of certain important natural objects, in continuum mechanics *ideal materials* are defined by particular relations between the stress tensor and the motion of the body. Some materials are important in themselves, but most of them are of more interest as members of a class than in detail. Thus a general theory of constitutive equations is needed. The material presented here draws heavily from the work of Noll.

3.2 Axioms for Constitutive Equations

3.2.1 *Principle of determinism*

The stress at the particle X in the body $\mathcal{B}$ at time t is determined by the history χ^t of the motion of $\mathcal{B}$ up to time t:

$$\mathbf{T}(X, t) = \mathcal{F}(\chi^t; X, t). \tag{3.2-1}$$

Here $\mathcal{F}$ is a functional in the most general sense of the term, namely a rule of correspondence. Equation (1) asserts that the motion of the body up to and including the present time determines a unique symmetric stress tensor $\mathbf{T}$ at each point of the body, and the manner in which it does so may depend upon X and t. The functional $\mathcal{F}$ is called the *constitutive functional*, and (1) is the *constitutive equation* of the ideal material defined by $\mathcal{F}$. Notice that the past, as much of it as necessary, may affect the present stress, but in general past and future are not interchangeable. The common prejudice that mechanics concerns phenomena reversible in time is too naive to need refuting.

3.2.2 *Principle of local action*

In the principle of determinism, the motions of particles Z that lie far away from X are allowed to affect the stress at X. The notion of contact force makes it natural to exclude action at a distance as a material property. Accordingly, we assume as a second axiom of continuum mechanics that the motion of particles at a finite distance from X in some configuration may be disregarded in calculating the stress at X. (Of course, by the smoothness assumed for χ, particles once a finite distance apart are always a finite distance apart.) Formally, if

$$\chi^{\bar{t}}(Z, s) = \chi^t(Z, s) \text{ when } s \geq 0 \text{ and } Z \in N(X), \tag{3.2-2}$$

where $N(X)$ is some neighborhood of X, then

$$\mathcal{F}(\chi^{\bar{t}}; X, t) = \mathcal{F}(\chi^t; X, t). \tag{3.2-3}$$

3.2.3 *Principle of material frame indifference*

We have said that we shall regard two equivalent dynamical processes as being really the same process, viewed by two different observers. We regard material properties as likewise indifferent to the choice of observer. Since constitutive equations are designed to express idealized material properties, we require them to be frame indifferent. That is, if (1) holds, namely,

$$\mathbf{T}(X, t) = \mathcal{F}(\chi^t; X, t), \tag{3.2-4}$$

then the constitutive functional $\mathcal{F}$ must be such that

$$\mathbf{T}(X, t^*)^* = \mathcal{F}[(\chi^{*t^*}); X, t^*], \tag{3.2-5}$$

where $\{\chi^*, T^*\}$ is any dynamical process equivalent to $\{\chi, \mathbf{T}\}$. The requirement of local action and frame indifference impose restrictions of the functionals $\mathcal{F}$ to be admitted in constitutive equations. Namely, $\mathcal{F}$ must be such that the constitutive equation (1) is invariant under the transformations (2.8-1). Only those functionals $\mathcal{F}$ that satisfy the requirements of local action and frame indifference are admissible as constitutive functionals.

Some steps may be taken to delimit the class of functionals satisfying these axioms, but here we shall treat only a special case, which is still general enough to include all the older theories of continua and most of the more recent ones. The special case is called the *simple material.*

3.3 Simple Materials

A motion χ is *homogeneous* with respect to the reference configuration κ if

$$\mathbf{x} = \chi_\kappa(\mathbf{X}, t) = \mathbf{F}(t)(\mathbf{X} - \mathbf{X}_0) + \mathbf{x}_0(t), \tag{3.3-1}$$

where $\mathbf{x}_0(t)$ is a place in χ, possibly moving, $\mathbf{X}_0$ is a fixed place in the reference configuration κ, and $\mathbf{F}(t)$, which is the deformation gradient, is a nonsingular tensor that does not depend on $\mathbf{X}$. A motion is homogeneous if and only if it carries every straight line at time 0 into a straight line at time t. A motion homogeneous with respect to one reference configuration generally fails to be so with respect to another.

Physically minded people almost always assume that everything there is to know about a material can be found out by performing experiments on homogeneous motions of a body of that material, from whatever state they happen to find it in. The materials in the special class that conforms to their prejudices are called *simple materials*. Formally, the material defined by (3.2-4) is called *simple* if there exists a reference configuration κ such that

$$\mathbf{T}(X, t) = \mathcal{F}(\chi^t; X, t) = \mathcal{G}_\kappa[\mathbf{F}^t(\mathbf{X}, s); X]. \tag{3.3-2}$$

That is, the stress at the place $\mathbf{x}$ occupied by the particle X at time t is determined by the totality of deformation gradients with respect to κ experienced by that particle up to the present time. $\mathcal{G}_\kappa$ is called a *response functional* of the simple material. Ordinarily we shall write (2) in one of the simpler forms

$$\mathbf{T} = \mathcal{G}_\kappa[\mathbf{F}^t(s)] = \mathcal{G}_\kappa(\mathbf{F}^t) = \mathcal{G}(\mathbf{F}^t), \tag{3.3-3}$$

with exactly the same meaning.

The defining equation (3) asserts that a given particle $\mathcal{F}$ depends on the motion χ^t only through its deformation gradient $\mathbf{F}^t$ with respect to some fixed reference configuration. That is, all motions having the same gradient history at a given particle and time give rise to the same stress at that particle and time. Since, trivially, a homogeneous motion with any desired gradient history may be constructed, only homogeneous motions need be considered in determining any material properties that the original constitutive functional $\mathcal{F}$ may describe.

We remark first that it is unnecessary to mention a particular reference configuration in the definition. In Chapter 1 we obtained the important formula

$$\mathbf{F}_1 = \mathbf{F}_2\mathbf{P} \tag{1.6-5}$$

connecting the gradients with respect to two reference configurations κ_1 and κ_2. By (3), then,

$$\mathbf{T} = \mathcal{G}_{\kappa_1}(\mathbf{F}_1^t) = \mathcal{G}_{\kappa_1}(\mathbf{F}_2^t\mathbf{P}). \tag{3.3-4}$$

Since $\mathbf{P}$ is the gradient of the transformation from κ_1 to κ_2, it is constant in time, and the right-hand side of (4) gives $\mathbf{T}$ as a functional of the history $\mathbf{F}_2^t$ of the deformation gradient with respect to κ_2. Thus, if we write κ in (3) as κ_1, and if we set

$$\mathcal{G}_{\kappa_2}(\mathbf{F}^t) = \mathcal{G}_{\kappa_1}(\mathbf{F}^t\mathbf{P}), \tag{3.3-5}$$

we see that a relation of the form (3) holds again if we take κ_2 as reference. Therefore, we may speak of a simple material without mentioning any particular reference configuration, and usually we do not write the subscript κ on $\mathcal{G}$. We must recall, however, that for a given simple material with constitutive functional $\mathcal{F}$ there are infinitely many different response functionals $\mathcal{G}_\kappa$, one for each choice of reference configuration κ.

Next we remark that, trivially, the simple material satisfies the principles of determinism and local action, no matter the response functional $\mathcal{G}$. It is not so for the principle of material frame indifference, as we shall see in the next section.

The theory of simple materials includes most of the common theories of continua studied in works on engineering, physics, applied mathematics, and so on. For example, the *elastic material* is defined by the special case when the functional $\mathcal{G}$ reduces to a function $\mathbf{g}$ of the present deformation gradient $\mathbf{F}(\mathbf{X}, t)$:

$$\mathbf{T} = \mathbf{g}(\mathbf{F}, \mathbf{X}), \tag{3.3-6}$$

where $\mathbf{g}$ is a function. The *linearly viscous material* is defined by the slightly more general case when $\mathcal{G}$ reduces to a function of $\mathbf{F}(t)$ and $\dot{\mathbf{F}}(t)$ that is linear in $\dot{\mathbf{F}}$:

$$\mathbf{T} = \mathbf{K}(\mathbf{F}, \mathbf{X})[\dot{\mathbf{F}}] = \mathbf{L}(\mathbf{F}, \mathbf{X})[\mathbf{G}], \tag{3.3-7}$$

where the second form follows from the first by (2.2-5). The Boltzmann accumulative theory is obtained by supposing that $\mathcal{G}$ is expressible as an integral. It is customary to restrict these theories still further by assuming that $|\mathbf{F} - \mathbf{1}|$ is small or imposing requirements of material symmetry, or both, as we shall see. Many but not all the recent nonlinear theories are included as special cases in the theory of simple materials. The simple material represents, in general, a material with long-range memory, so stress relaxation, creep, and fatigue can occur.

EXERCISE 3.3.1

Prove that (7) satisfies the principle of material frame indifference if and only if it is equivalent to

$$\mathbf{R}^T\mathbf{T}\mathbf{R} = \mathbf{L}(\mathbf{U}, \mathbf{X})[\mathbf{R}^T\mathbf{D}\mathbf{R}], \tag{3.3-8}$$

where $\mathbf{L}$ is a linear operator.

3.4 Restriction for Material Frame Indifference

Under a change of frame, $\mathbf{F}^* = \mathbf{QF}$, as we saw in Chapter 2. Let $\mathbf{Q}^t(s)$ be the history up to time t of any orthogonal tensor function $\mathbf{Q}(t)$. Then the principle of material frame indifference states that the response functional $\mathcal{G}$ of a simple material must satisfy the equation

$$\mathcal{G}_\kappa[\mathbf{Q}^t(s)\mathbf{F}^t(s)] = \mathbf{Q}(t)\mathcal{G}_\kappa[\mathbf{F}^t(s)]\mathbf{Q}(t)^T \tag{3.4-1}$$

for every orthogonal tensor history $\mathbf{Q}^t(s)$ and every nonsingular tensor history $\mathbf{F}^t(s)$.

We can solve this equation. According to the polar decomposition theorem, $\mathbf{F}^t(s) = \mathbf{R}^t(s)\mathbf{U}^t(s)$, so

$$\mathcal{G}_\kappa[\mathbf{F}^t(s)] = \mathbf{Q}(t)^T\mathcal{G}_\kappa[\mathbf{Q}^t(s)\mathbf{R}^t(s)\mathbf{U}^t(s)]\mathbf{Q}(t). \tag{3.4-2}$$

Since this equation must hold for all $\mathbf{Q}^t$, $\mathbf{R}^t$, $\mathbf{U}^t$, it must hold in particular if $\mathbf{Q}^t = (\mathbf{R}^t)^T$. Therefore

$$\mathcal{G}_\kappa(\mathbf{F}^t) = \mathbf{R}(t)\mathcal{G}_\kappa(\mathbf{U}^t)\mathbf{R}(t)^T. \tag{3.4-3}$$

Conversely, suppose $\mathcal{G}_\kappa$ is of this form, and consider an arbitrary orthogonal tensor history $\mathbf{Q}^t$. Since the polar decomposition of $\mathbf{Q}^t\mathbf{F}^t$ is $(\mathbf{Q}^t\mathbf{R}^t)\mathbf{U}^t$,

$$\begin{aligned}\mathcal{G}_\kappa(\mathbf{Q}^t\mathbf{F}^t) &= \mathbf{Q}(t)\mathbf{R}(t)\mathcal{G}_\kappa(\mathbf{U}^t)[\mathbf{Q}(t)\mathbf{R}(t)]^T \\ &= \mathbf{Q}(t)\mathcal{G}_\kappa(\mathbf{F}^t)\mathbf{Q}(t)^T,\end{aligned} \tag{3.4-4}$$

so that (1) is satisfied. Therefore, (3) gives the general solution of the functional equation (1). We have proved then, that the *constitutive equation of a simple material may be put into the form*

$$\mathbf{T} = \mathbf{R}\mathcal{G}_\kappa(\mathbf{U}^t)\mathbf{R}^T \tag{3.4-5}$$

and, conversely, that any functional $\mathcal{G}_\kappa$ of positive-definite symmetric tensor histories, if its values are symmetric tensors, *serves to define a simple material through* (5). A constitutive equation of this kind, in which the functionals or functions occurring are not subject to any further restriction, is called a *reduced form.*

The result (5) shows us that while the stretch history $\mathbf{U}^t$ of a simple material may influence its present stress in any way whatever, past rotations have no influence at all. The present rotation $\mathbf{R}$ enters (5) explicitly. Thus, the reduced form enables us to dispense with considering rotation in determining the response of a material. If we like, we may regard (3) as effecting an extension of $\mathcal{G}_\kappa$ from the range of positive-definite symmetric tensor histories to the full range of nonsingular tensor histories.

The reduced form enables us also, in principle, to reduce the number of tests needed to determine the response functional $\mathcal{G}_\kappa$ by observation. Indeed, consider pure stretch histories: $\mathbf{R}^t = \mathbf{1}$. If we know the stress $\mathbf{T}$ corresponding to an arbitrary homogeneous pure stretch history $\mathbf{U}^t$, we have a relation of the form $\mathbf{T} = \mathcal{G}_\kappa(\mathbf{U}^t)$. By (5) we then know $\mathbf{T}$ for all deformation histories. Alternatively, consider irrotational histories: $\mathbf{W} = \mathbf{0}$. Given any $\mathbf{U}^t$, we can determine $\mathbf{R}^t$ by integrating $(2.2\text{-}11)_1$ with $\mathbf{W}$ set equal to $\mathbf{0}$. If we know the stress corresponding to an arbitrary irrotational history, by putting the corresponding values of $\mathbf{R}$ into (5) we can again determine $\mathcal{G}_\kappa$. Thus, we may characterize simple materials in either of two more economical ways: *A material is simple if and only if its response to all deformation is determined by its response to all homogeneous pure stretch histories or to all homogeneous irrotational deformation histories.*

In the polar decomposition (2.1-1), two measures of stretch, $\mathbf{U}$ and $\mathbf{V}$, are introduced. Kinematically, there is no reason to prefer one to the other. From (3) we see that use of $\mathbf{U}$ to measure the stretch history leads to a simple reduced form for the constitutive equation of a simple material. If we like, of course we may use $\mathbf{V}$ instead. Since $\mathbf{U}^t = (\mathbf{R}^t)^T\mathbf{V}^t\mathbf{R}^t$, substitution in (3) shows that by using $\mathbf{V}$ we do not generally eliminate the rotation history $\mathbf{R}^t$. That is, use of $\mathbf{V}$ does not lead to a simple result. There are many other tensors that measure stretch just as well as $\mathbf{U}$ and $\mathbf{V}$. In the older literature one or another of these is called a "strain"tensor. We prefer not to use the term "strain."

There are infinitely many other reduced forms for the constitutive equation of a simple material. Since $\mathbf{C}^t = (\mathbf{U}^t)^2$, one such form is

$$\mathbf{T} = \mathbf{R}\mathbf{U}\mathbf{U}^{-1}\mathcal{G}(\sqrt{\mathbf{C}}^t)\mathbf{U}^{-1}\mathbf{U}\mathbf{R}^T = \mathbf{F}\mathcal{L}(\mathbf{C}^t)\mathbf{F}^T, \tag{3.4-6}$$

where

$$\mathcal{L}(\mathbf{C}^t) \equiv \mathbf{U}^{-1}\mathcal{G}(\sqrt{\mathbf{C}}^t)\mathbf{U}^{-1}.$$

EXERCISE 3.4.1
Using (1.7-7), derive *Noll's reduced form* for the constitutive equation of a simple material:

$$\bar{\mathbf{T}}(t) = \mathcal{A}[\bar{\mathbf{C}}_t^{-t}; \mathbf{C}(t)], \tag{3.4-7}$$

where for any tensor $\mathbf{K}$ the tensor $\bar{\mathbf{K}}$ is defined by

$$\bar{\mathbf{K}} = \mathbf{R}(t)^T \mathbf{K} \mathbf{R}(t). \tag{3.4-8}$$

This result (7) shows that it is not possible to express the effect of the deformation history on the stress entirely by measuring deformation with respect to the present configuration. While the effect of all the *past* history, $0 < s < \infty$, is accounted for in this way, a fixed reference configuration is required, in general to allow for the effect of the deformation at the present instant, as shown by the appearance of $\mathbf{C}(t)$ as a parameter in (7). The result itself is important in that it enables us to go as far as possible toward avoiding use of a fixed reference configuration. Roughly, it shows that memory effects can be accounted entirely by use of the relative deformation, but finite-strain effects require use of some fixed reference configuration, any one we please. This result should not surprise anyone, since in the theory of finite elastic strain the stress tensor is altogether independent of the relative deformation and hence cannot be expressed in terms of it.

3.5 Internal Constraints

So far, we have been assuming that the material is capable, if subjected to appropriate forces, of undergoing any smooth motion. If the class of possible motions is limited at interior points of $\mathcal{B}$, the material is said to be subject to an *internal constraint*. A *simple constraint* is expressed by requiring the deformation gradient $\mathbf{F}$ to satisfy an equation of the form

$$\gamma(\mathbf{F}) = 0, \tag{3.5-1}$$

where γ is a frame-indifferent scalar function.

EXERCISE 3.5.1
Prove that $\gamma(\mathbf{F})$ is frame indifferent if and only if

$$\gamma(\mathbf{F}) = \gamma(\mathbf{U}). \tag{3.5-2}$$

Hence a simple constraint may be written in the form

$$\lambda(\mathbf{C}) = 0, \tag{3.5-3}$$

where λ is a scalar function, and every such equation expresses a frame-indifferent simple constraint.

If we differentiate (3) with respect to time for a fixed particle, we obtain

$$\dot{\lambda} = \mathrm{tr}[\partial_{\mathbf{C}}\lambda(\mathbf{C})\dot{\mathbf{C}}] = 0. \tag{3.5-4}$$

But by (2.1-4) and (1.7-7),

$$\mathbf{C}(\tau) = \mathbf{F}(\tau)^T\mathbf{F}(\tau) = \mathbf{F}(t)^T\mathbf{C}_t(\tau)\mathbf{F}(t). \tag{3.5-5}$$

If we differentiate this relation with respect to τ and then put $\tau = t$, by $(2.2\text{-}2)_2$ we find that

$$\dot{\mathbf{C}}(t) = 2\mathbf{F}(t)^T\mathbf{D}(t)\mathbf{F}(t). \tag{3.5-6}$$

Hence (4) becomes

$$\mathrm{tr}[\mathbf{F}\partial_{\mathbf{C}}\lambda(\mathbf{C})\mathbf{F}^T\mathbf{D}] = 0 \tag{3.5-7}$$

for all nonsingular $\mathbf{F}$ and all symmetric $\mathbf{D}$. Conversely, if (7) holds at each instant for the particle in question, (4) follows from it by integration. Thus (7) may be used alternatively as a general expression for a frame-indifferent simple constraint.

3.6 Principle of Determinism for Constrained Simple Materials

Constraints, since they consist of the prevention of some kinds of motion, must be maintained by forces. Since the constraints, by definition, are immutable, the forces maintaining them cannot be determined by the motion itself or its history. In particular, simple internal constraints must be maintained by appropriate stresses, and the constitutive equation of a constrained simple material must allow these stresses to operate, regardless of the deformation history.

For constrained materials, accordingly, the principle of determinism must be relaxed. *A fortiori*, the necessary modification of that principle cannot be deduced from the principle itself but must be brought in as a new axiom.

There are, presumably, many systems of forces that could bring about any given constraint. The simplest of these are the ones that do no work in any motion compatible with the constraint. In a constrained material these forces will therefore be assumed to remain arbitrary in the sense that they are not determined by the history of the motion.

Thus we have given reasons for laying down the following *principle of determinism for simple materials subject to constraints*: The stress is determined by the history of the motion only to within an arbitrary tensor that does no work in any motion compatible with the constraint. That is,

$$\mathbf{T} = \mathbf{N} + \bar{\mathcal{G}}(\mathbf{F}^t), \tag{3.6-1}$$

where $\mathbf{N}$ does no work in any motion satisfying the constraint and where $\overline{\mathcal{G}}$ must be defined only for such arguments $\mathbf{F}^t$ as satisfy the constraint. $\mathbf{T} - \mathbf{N}$ is called the *determinate stress*.

The problem is now to find $\mathbf{N}$. The rate of working of a symmetric stress tensor $\mathbf{T}$ in a motion with stretching tensor $\mathbf{D}$ is the *stress working* (or stress power) w:

$$\mathrm{w} := \mathrm{tr}(\mathbf{TD}). \tag{3.6-2}$$

Accordingly, we are to find the general solution $\mathbf{N}$ of the equation

$$\mathrm{tr}(\mathbf{ND}) = 0, \tag{3.6-3}$$

where $\mathbf{D}$ is any symmetric tensor that satisfies (3.5-7). Now $\mathbf{F}\partial_\mathbf{C}\lambda(\mathbf{C})\mathbf{F}^T$ is a symmetric tensor, and the operation $\mathrm{tr}(\mathbf{AB})$ defines an inner product in the space of symmetric tensors. Hence $\mathbf{N}$, regarded as a vector, must be perpendicular to every vector $\mathbf{D}$ that is perpendicular to $\mathbf{F}\partial_\mathbf{C}\lambda(\mathbf{C})\mathbf{F}^T$. Thus $\mathbf{N}$ is parallel to the latter vector:

$$\mathbf{N} = q\mathbf{F}\partial_\mathbf{C}\lambda(\mathbf{C})\mathbf{F}^T, \tag{3.6-4}$$

where q is a scalar function of $\mathbf{F}$.

If there are k constraints $\lambda^i(\mathbf{C}) = 0$, then

$$\mathbf{N} = \sum_{i=1}^{\mathrm{k}} \mathrm{q_i}\mathbf{F}\partial_\mathbf{C}\lambda^i(\mathbf{C})\mathbf{F}^T. \tag{3.6-5}$$

Substitution into (3.6-1) yields the general constitutive equation for simple material subject to k simple frame-indifferent internal constraints.

The determinate part of the stress, $\mathcal{G}_\kappa(\mathbf{F})^t$, may be expressed in reduced forms identical with those found for unconstrained materials.

3.7 Examples of Internal Constraints

3.7.1 *Incompressibility*

A material is said to be *incompressible* if it can experience only isochoric motions. By $(1.4\text{-}6)_3$ and $(2.1\text{-}6)_9$, an appropriate constraint function is

$$\lambda(\mathbf{C}) = \det\mathbf{C} - 1. \tag{3.7-1}$$

Since

$$\mathbf{F}\partial_\mathbf{C}\lambda(\mathbf{C})\mathbf{F}^T = \mathbf{F}\mathbf{C}^{-1}\mathbf{F}^T, \det\mathbf{C} = 1, \tag{3.7-2}$$

(3.6-4) yields

$$\mathbf{N} = -p\mathbf{1}, \tag{3.7-3}$$

where p is an arbitrary scalar. Thus we have verified a result due in effect to Poincaré: In an incompressible material, the stress is determinate from the motion only to within an arbitrary hydrostatic pressure. For future reference we shall express the constitutive relation for an incompressible material through

$$\mathbf{T} = -p\mathbf{1} + \mathbf{S}, \qquad \mathbf{S} = \mathcal{G}[\mathbf{F}^t]; \tag{3.7-4}$$

the response $\mathcal{G}$ need not be defined except for arguments such that $|\det \mathbf{F}^t| = 1$. Also, we shall define

$$\varphi := \frac{p}{\rho} + \varpi. \tag{3.7-5}$$

3.7.2 *Inextensibility*

If $\mathbf{e}$ is a unit vector in the reference configuration, $\mathbf{Fe}$ is the vector into which it is carried in a homogeneous deformation with gradient $\mathbf{F}$. Accordingly, for a material *inextensible* in the direction $\mathbf{e}$, an appropriate constraint function is

$$\lambda(\mathbf{C}) = (\mathbf{Fe})^2 - 1 = \mathbf{e} \cdot \mathbf{Ce} - 1. \tag{3.7-6}$$

Since

$$\partial_{\mathbf{C}}\lambda(\mathbf{C}) = \mathbf{e} \otimes \mathbf{e}, \tag{3.7-7}$$

(3.6-4) yields

$$\mathbf{N} = q\mathbf{F}(\mathbf{e} \otimes \mathbf{e})\mathbf{F}^T = q\mathbf{Fe} \otimes \mathbf{Fe}. \tag{3.7-8}$$

Since $\mathbf{N}$ is an arbitrary uniaxial tension in the direction of $\mathbf{Fe}$, we recover a result first found by Adkins and Rivlin: in a material inextensible in a certain direction, the stress is determinate only to within a uniaxial tension in that direction.

3.7.3 *Rigidity*

A material is rigid if it is inextensible in every direction. By the result just established, the stress in a rigid material is determinate only to within an arbitrary tension in every direction. In other words, the stress in a rigid material is altogether unaffected by the motion.

EXERCISE 3.7.1

Prove that the conclusions of the foregoing three examples hold for every possible choice of the constraint function λ.

3.8 Importance of Constrained Materials

We have just made it plain that a constrained material is by no means a special case of an unconstrained one. Rather, the reverse holds, and the unconstrained material emerges as special. The behavior of a constrained material is not the same as that of an unconstrained one that happens to experience a motion satisfying the constraint. For example, if an unconstrained material happens to be subjected to an isochoric deformation history, its stress is determined by that history. An incompressible material, by definition, can never be subjected to anything but isochoric deformation histories, but its stress is never completely determined by them, because it is always indeterminate to the extent of an arbitrary hydrostatic pressure. Hydrodynamic writers are guilty of propagating confusion when they refer to "incompressible flows." A flow, obviously, cannot be compressed. It may or may not be isochoric, and a fluid may or may not be incompressible; the behavior of a compressible fluid in an isochoric flow is generally not at all the same as that of an incompressible fluid in the same flow.

A constrained material is susceptible to a *smaller* class of deformations. Corresponding to this restriction, arbitrary stresses arise. Their presence makes a *greater variety* of response possible in those deformation that do occur. Consequently, solution of problems becomes *easier*. This is reflected in the far fewer exact solutions that are available for both compressible fluids and solids in comparison to their incompressible counterparts.

The extreme case is furnished by the rigid material, where the deformations allowed are reduced to so special a class that the stress plays no part at all in the motion, which can be determined by solving ordinary differential equations that express no more than the principles of momentum and moment of momentum for the body as a whole.

The most useful case is that of the incompressible material. For a simple material in general, substitution of (3.3-3) into Cauchy's first law (2.7-5) yields, when $\mathbf{b} = \mathbf{0}$,

$$\operatorname{div} \mathcal{G}(\mathbf{F}^t) = \rho \ddot{\mathbf{x}}, \tag{3.8-1}$$

while for an incompressible material use of (3.6-1) and (3.7-3) in the same way yields

$$-\operatorname{grad} p + \operatorname{div} \mathcal{G}(\mathbf{F}^t) = \rho \ddot{\mathbf{x}}. \tag{3.8-2}$$

These equations must be satisfied by any deformation history that can be produced by surface tractions alone. For the former, all $\mathbf{F}^t$ are eligible to compete, and few will be found successful. For the latter, only those $\mathbf{F}^t$ such that $\det \mathbf{F}^t = 1$ are allowed, but *p may be adjusted* to aid in finding a solution. The condition upon the motion alone, of course, is now

$$\operatorname{curl}\operatorname{div}\mathcal{G}(\mathbf{F}^t) = \operatorname{curl}(\rho\ddot{\mathbf{x}}), \tag{3.8-3}$$

a differential equation of higher order than (1).

Next we document "the vorticity equation" that we shall appeal to repeatedly later. It follows from (3.7-4) and Cauchy's first law that

$$\rho(\ddot{\mathbf{x}} - \mathbf{b}) = -\operatorname{grad} p + \operatorname{div}\mathcal{G}(\mathbf{F}^t). \tag{3.8-4}$$

If a field p satisfies this equation, so also does $p + h$ for an arbitrary function h of t alone. This is in effect because a uniform pressure acting on the boundary of an incompressible fluid exerts no resultant force or torque on that body.

If the body force $\mathbf{b}$ is conservative, then from (2.2-15), (2.2-16), and (4), we obtain a useful form of integrability necessary and sufficient for φ to exist:

$$\rho\mathbf{W}_a = \rho(\dot{\mathbf{W}} + \mathbf{D}\mathbf{W} + \mathbf{W}\mathbf{D}) = \operatorname{skw}\operatorname{grad}\operatorname{div}\mathcal{G}(\mathbf{F}^t). \tag{3.8-5}$$

EXERCISE 3.8.1

Show that for a motion of an incompressible body whose response is $\mathcal{G}$, to preserve circulation, it is necessary and sufficient that

$$\operatorname{skw}\operatorname{grad}\operatorname{div}\mathcal{G}\left(\mathbf{F}^t\right) = 0, \tag{3.8-6}$$

and hence that during the motion there is a scalar field λ such that

$$\operatorname{div}\mathcal{G}(\mathbf{F}^t) = -\operatorname{grad}\ \lambda; \tag{3.8-7}$$

thus

$$p = \rho(P_a - \varpi) - \lambda. \tag{3.8-8}$$

EXERCISE 3.8.2 (Coleman and Truesdell)

For a homogeneous, incompressible body whose respsone is $\mathcal{G}$, let a flow that preserves circulation be possible subject to null body force. Show that the stress can be expressed as

$$\mathbf{T} = -[\rho(P_a - \varpi) - \lambda]\mathbf{1} + \mathcal{G}(\mathbf{F}^t), \tag{3.8-9}$$

where P_a is the acceleration potential of the flow and ϖ is given by (2.6-3).

Nearly all the exact solutions found in nonlinear continuum theories are for incompressible materials.

4

Simple Fluids

4.1 Definition of a Simple Fluid

There are various physical notions concerned with fluids. One is that a fluid is a substance that can flow. "Flow" itself is a vague term. One meaning of "flow" is simply deformation under stress, which does not distinguish a fluid from any other nonrigid material. Another is that steady velocity results from constant stress, which seems to be special and to apply only with difficulty and to particular flows. Another is inability to support shear stress when in equilibrium Formally, within the theory of simple materials, such a definition would yield

$$\mathbf{T} = -p(\rho)\mathbf{1} + \mathcal{F}(\mathbf{F}^t), \tag{4.1-1}$$

where $\mathcal{F}(\mathbf{1}^t) = \mathbf{0}$, $\mathbf{1}^t$ being the history whose value is always $\mathbf{1}$.

Fundamental Theorem on Fluids[1]. *Every fluid has a constitutive relation of the form*

$$\mathbf{T} = \mathcal{R}(\mathbf{C}_t^t; \rho), \tag{4.1-2}$$

and

$$\mathcal{R}(\mathbf{Q}\mathbf{C}_t^t\mathbf{Q}^t; \rho) = \mathbf{Q}\mathcal{R}(\mathbf{C}_t^t; \rho)\mathbf{Q}^T \tag{4.1-3}$$

for all orthogonal $\mathbf{Q}$ *and all arguments* $\mathbf{C}_t^t$, ρ *that lie in the domain of* $\mathcal{R}$. *Every such isotropic mapping of positive symmetric tensor histories into symmetric tensors defines a fluid. Furthermore,*

$$\mathcal{R}(\mathbf{1}^t; \rho) = -p(\rho)\mathbf{1}. \tag{4.1-4}$$

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[1] A proof of the this theorem can be found in Truesdell's *First Course in Rational Continuum Mechanics*," vol. 1 (New York: Academic Press, 1991).

This result states that all fluids obey in equilibrium the laws of Eulerian hydrostatics, according to which the stress is a hydrostatic pressure that depends on the density alone. In particular, a fluid exhibits the phenomenon of "flow" in one of the common senses, namely, that it cannot support any shear stress when it has been at rest for all times, past and present, in any placement whatever.

We may also express the foregoing result as follows: The constitutive relation of a fluid is of the form

$$\mathbf{T} = -p(\rho)\mathbf{1} + \mathcal{C}(\mathbf{C}_t^t - \mathbf{1}; \rho); \tag{4.1-5}$$

the mapping $\mathcal{C}$ is isotropic, and it vanishes when its argument is the history $\mathbf{0}^t$ whose value is always $\mathbf{0}$. Conversely, every relation of this form defines a fluid.

While in hydrodynamics it is customary to impose the condition that $p(\rho) > 0$ for all ρ, or at least the weaker requirement that $p(\rho) > 0$ for all but a discrete set of values of p, this condition does not follow from any general principle of mechanics. Since hydrostatic tensions of some magnitude have been produced, with extreme pains, in certain very quiet laboratories, perhaps the condition $p(\rho) > 0$ should be regarded as expressing stability rather than as a constitutive restriction.

A fluid may react to its entire deformation history, yet its reaction cannot be different for different placements with the same density. A fluid reconciles these two seemingly contradictory qualities—ability to remember all its past and inability to regard one placement as different from another—by reacting to the past only insofar as it may differ from the present, which may be ever changing.

An important simple fluid that occupies a central place in fluid mechanics is the linearly viscous fluid, or Navier-Stokes fluid, whose Cauchy stress takes the form

$$\mathbf{T} = (-p + \lambda \operatorname{tr} \mathbf{D})\mathbf{1} + 2\mu\mathbf{D}, \tag{4.1-6}$$

in which p, λ and μ are functions of ρ. In the case of an incompressible linearly viscous fluid, it follows from (3.7-4) and (6) that

$$\mathbf{T} = -p\mathbf{1} + 2\mu\mathbf{D}, \tag{4.1-7}$$

where p is indeterminate in the sense that it may be assigned any value independent of the history of the motion.

A special subclass of fluids of the form (6) is the unconstrained elastic fluid that has the constitutive relation

$$\mathbf{T} = -p(\rho)\mathbf{1}, \tag{4.1-8}$$

and the incompressible elastic fluid is defined through

$$\mathbf{T} = -p\mathbf{1}, \tag{4.1-9}$$

where p is indeterminate.

4.2 Monotonous Motions

Continuum mechanics, even the mechanics of simple materials, covers so vast a range of possible behavior that little can be learned from it without studying special cases. In this complexity, continuum mechanics mirrors nature itself, for only by specifying particular features of a phenomenon can we so much as name it, let alone describe it. In the mechanics of simple materials, two kinds of specialization are fruitful: specialization of the material and specialization of the motions to which the body is subjected. We have given examples of the former in the immediately preceding sections. The constitutive relations of fluids and isotropic solids are simpler than the general one, and we can expect the solution of problems for these two classes of materials to be relatively easier than for anisotropic solids or fluid crystals. The continuum mechanics of the last century carried this kind of specialization much further and restricted attention to materials specified by one or two constants. As a result, the solution of wide classes of boundary-value problems became easy—deceptively so, since only rarely can the properties of natural bodies be condensed adequately into one or two numbers fit to be tabulated in a handbook.

Consider, for example, the constitutive equation of a simple fluid:

$$\mathbf{T} = \mathcal{R}(\mathbf{C}_t^t, \rho). \tag{4.1-2}$$

In the particular case when ρ = const. and $\mathbf{C}_t^t(s)$ is the same function of s for all t, the stress becomes constant in time for a given particle. The body may have experienced deformation for all past time, but as each body point looks backward, so to speak, it sees the entire sequence of past deformations referred to its present placement remain unchanged.

More generally, since the principle of material frame indifference forbids past rotations to enter the constitutive relation and renders explicit the effect of present rotation, we should be able to simplify the constitutive relation almost as much in the somewhat more general case when for some orthogonal tensor $\mathbf{Q}(t)$

$$\mathbf{C}_t^t(s) = \mathbf{Q}(t)\mathbf{C}_0^0(s)\mathbf{Q}(t)^T, \qquad 0 \le s < \infty. \tag{4.2-1}$$

Here, of course, $\mathbf{C}_0^0$ denotes $\mathbf{C}_t^t$ when $t = 0$ and $\mathbf{Q}(0) = \mathbf{1}$. In such motions an observer situated upon the moving particle may choose his frame in such a way as to see behind him always the same deformation history referred to the present placement. The proper numbers of $\mathbf{C}_t^t(s)$ are the same as those of $\mathbf{C}_0^0(s)$, although the principal axes of the one tensor may rotate arbitrarily with respect to those of the other. Thus, while the principal relative stretches $v_{(t)i}$ may vary with t, they do so in such a way that their histories up to the time t remain unchanged:

$$v_{(t)k}^t = v_{(0)k}^0, \quad k = 1, 2, 3, \quad -\infty < t < \infty. \tag{4.2-2}$$

We turn now to the pure kinematics of motions with constant principal relative stretch histories. All such motions are characterized by the following Theorem.

FUNDAMENTAL THEOREM (Noll). *A motion is monotonous if and only if there are an orthogonal tensor* $\mathbf{Q}(t)$*, a scalar* κ*, and a constant tensor* $\mathbf{N}_0$ *such that*

$$\mathbf{F}_0(\tau) = \mathbf{Q}(\tau)e^{t\kappa N_0}, \mathbf{Q}(0) = \mathbf{1}, \quad |\mathbf{N}_0| = 1. \tag{4.2-3}$$

PROOF We begin from the hypothesis (1) and set

$$\mathbf{H}(s) = \mathbf{C}_0(-s) = \mathbf{Q}(t)^T\mathbf{C}_t(t-s)\mathbf{Q}(t). \tag{4.2-4}$$

By (1.7-8), $\mathbf{F}_t(\tau) = \mathbf{F}_0(\tau)\mathbf{F}_0(t)^{-1}$, so

$$\begin{aligned}\mathbf{Q}(t)\mathbf{H}(s)\mathbf{Q}(t)^T &= \mathbf{C}_t(t-s)\\ &= [\mathbf{F}_0(t)^T]^{-1}\mathbf{C}_0(t-s)\mathbf{F}_0(t)^{-1}\\ &= [\mathbf{F}_0(t)^T]^{-1}\mathbf{H}(s-t)\mathbf{F}_0(t)^{-1}.\end{aligned} \tag{4.2-5}$$

If we set

$$\mathbf{E}(t) := \mathbf{Q}(t)^T\mathbf{F}_0(t), \tag{4.2-6}$$

then (5) assumes the form of a difference equation:

$$\mathbf{H}(s-t) = \mathbf{E}(t)^T\mathbf{H}(s)\mathbf{E}(t). \tag{4.2-7}$$

To obtain a necessary condition for a solution $\mathbf{H}(s)$, we differentiate[2] (7) with respect to t and put $t = 0$, obtaining the first-order linear differential equation

$$-\dot{\mathbf{H}}(s) = \mathbf{M}^T\mathbf{H}(s) + \mathbf{H}(s)\mathbf{M}; \tag{4.2-8}$$

here $\mathbf{M} \equiv \dot{\mathbf{E}}(0)$, and the dot denotes differentiation with respect to s. The unique solution of (8) such that $\mathbf{H}(0) = \mathbf{1}$ is easily seen to be

$$\mathbf{H}(s) = e^{-s\mathbf{M}^T}e^{-s\mathbf{M}}. \tag{4.2-9}$$

Since histories are defined only when $s \geq 0$, this result has been derived only for that domain. Nevertheless, the difference equation (7) serves to define $\mathbf{H}(s)$ for negative s as well and shows that $\mathbf{H}$ is analytic. Since the right-hand side of (9) is analytic, the principle of analytic continuation shows that (9) gives the unique solution for all s, when $\mathbf{E}(t)$ is assigned. If we substitute (9) back into (7), by putting $s = 0$ we obtain

$$\mathbf{E}(t)e^{-t\mathbf{M}}[\mathbf{E}(t)e^{-t\mathbf{M}}]^T = \mathbf{1}. \tag{4.2-10}$$

Hence $\mathbf{E}(t)e^{-t\mathbf{M}}$ is an orthogonal tensor, say $\bar{\mathbf{Q}}(t)$. By (6), then

$$\mathbf{F}_0(t) = \mathbf{Q}(t)\bar{\mathbf{Q}}(t)e^{t\mathbf{M}}. \tag{4.2-11}$$

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[2]That the assertion of the theorem remains true even if **H** is merely continuous and **E** is completely arbitrary has been shown by W. Noll, "The representation of monotonous processes by exponentials," *Indiana University Mathematics Journal* 25, (1976): 209–14.

The form asserted by Noll's theorem holds trivially if $\mathbf{M} = \mathbf{0}$; if $\mathbf{M} \neq \mathbf{0}$, it follows if we set

$$\kappa := |\mathbf{M}|, \qquad \mathbf{N}_0 = \frac{1}{\kappa}\mathbf{M}. \tag{4.2-12}$$

(The proof reveals that the tensor $\mathbf{Q}(t)$ occurring in the result (3) is not generally the same orthogonal tensor as that occurring in the hypothesis (1).) Conversely, if (3) holds, an easy calculation shows that the motion is one with constant relative principal stretch histories.

EXERCISE 4.2.1 (Noll)
Prove that in a monotonous motion

$$\mathbf{F}_t(\tau) = \mathbf{Q}(\tau)\mathbf{Q}(t)^T e^{(\tau-t)\kappa\mathbf{N}} = \mathbf{Q}(\tau)e^{(\tau-t)\kappa\mathbf{N}_0}\mathbf{Q}(t)^T, \tag{4.2-13}$$

with $\mathbf{N}$ defined as follows:

$$\mathbf{N} = \mathbf{Q}(t)\mathbf{N}_0\mathbf{Q}(t)^T, \quad |\mathbf{N}| = 1; \tag{4.2-14}$$

conversely, if $\mathbf{F}_t(\tau)$ has the form (13), any motion to which it corresponds is a motion with constant principal relative stretch histories. In such a motion

$$\begin{aligned}
\mathbf{C}_t^t(s) &= e^{-s\kappa\mathbf{N}^T}e^{-s\kappa\mathbf{N}},\\
\mathbf{G} &= \kappa\mathbf{N} + \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T,\\
\mathbf{A}_1 &= \dot{\mathbf{C}}_t^t(0) = \kappa(\mathbf{N} + N^T),\\
\mathbf{A}_2 &= \ddot{\mathbf{C}}_t^t(0) = \kappa(\mathbf{N}^T\mathbf{A}_1 + \mathbf{A}_1\mathbf{N}) = \kappa^2(2\mathbf{N}^T\mathbf{N} + \mathbf{N}^2 + (\mathbf{N}^T)^2),\\
\mathbf{A}_3 &= \kappa(\mathbf{N}^T\mathbf{A}_2 + \mathbf{A}_2\mathbf{N}), \ldots,\\
\mathbf{A}_k &= \kappa(\mathbf{N}^T\mathbf{A}_{k-1} + \mathbf{A}_{k-1}\mathbf{N}),
\end{aligned} \tag{4.2-15}$$

with the notations of 2.2. A monotonous motion is isochoric if and only if

$$\operatorname{tr}\mathbf{N}_0 = 0 \tag{4.2-16}$$

and of course also $\operatorname{tr}\mathbf{N} = 0$.

With the aid of these results we are able to see easily the extremely special nature of monotonous motions, which is expressed by the following corollary.

COROLLARY (Wang). *The relative deformation history* $\mathbf{C}_t^t$ *of a monotonous motion is determined uniquely by its first three Rivlin-Ericksen tensors.*

That is, if three tensors $\mathbf{A}_1(t)$, $\mathbf{A}_2(t)$, and $\mathbf{A}_3(t)$ are given, they can be the first three Rivlin-Ericksen tensors corresponding to at most one relative deformation history $\mathbf{C}_t^t$ satisfying the defining condition (1).

The proof rests upon a simple lemma. Let **S** be a symmetric tensor and **W** a skew tensor in three-dimensional space. Without loss of generality we can take the matrices of these tensors as having the forms

$$[\mathbf{S}] = \begin{Vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{Vmatrix}, \qquad [\mathbf{W}] = \begin{Vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{Vmatrix}. \tag{4.2-17}$$

Then

$$[\mathbf{SW} - \mathbf{WS}] = \begin{Vmatrix} 0 & (a-b)x & (a-c)y \\ (a-b)x & 0 & (b-c)z \\ (a-c)y & (b-c)z & 0 \end{Vmatrix}. \tag{4.2-18}$$

Hence **S** and **W** commute if and only if

$$(a-b)x = 0, \quad (a - cy = 0), \quad (b-c)z = 0. \tag{4.2-19}$$

Consequently, if **S** has three proper numbers, it commutes with no skew tensor other than **0**. If $a = b \neq c$, **S** commutes with **W** if and only if $y = z = 0$. If $a = b = c$, **S** commutes with all **W**.

Wang's corollary may now be proved in stages. If two motions with constant principal relative stretch histories can correspond to $\mathbf{A}_1$ and $\mathbf{A}_2$, then because of $(15)_{4,6}$ there are tensors **M** and $\bar{\mathbf{M}}$ such that

$$\begin{aligned} \mathbf{M} + \mathbf{M}^T &= \bar{\mathbf{M}} + \bar{\mathbf{M}}^T, \\ \mathbf{M}^T\mathbf{A}_1 + \mathbf{A}_1\mathbf{M} &= \bar{\mathbf{M}}^T\mathbf{A}_1 + \mathbf{A}_1\bar{\mathbf{M}}. \end{aligned} \tag{4.2-20}$$

The first of these equations asserts that $\mathbf{M} - \bar{\mathbf{M}}$ is skew, the second that $\mathbf{M} - \bar{\mathbf{M}}$ commutes with $\mathbf{A}_1$. If $\mathbf{A}_1$ has three proper numbers, the lemma shows that $\mathbf{M} - \bar{\mathbf{M}} = \mathbf{0}$.

Suppose now that $\mathbf{A}_1$ has two proper numbers. Then relative to a suitable orthonormal basis,

$$[\mathbf{A}_1] = \begin{Vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{Vmatrix}, \quad a \neq b. \tag{4.2-21}$$

Case 1. Assume that, relative to this same basis,

$$[\mathbf{A}_2] = \begin{Vmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{Vmatrix}. \tag{4.2-22}$$

The most general **M** compatible with $(15)_4$ and (21) is given by

$$\kappa[\mathbf{M}] = \begin{Vmatrix} \frac{1}{2}a & x & y \\ -x & \frac{1}{2}a & z \\ -y & -z & \frac{1}{2}b \end{Vmatrix}. \tag{4.2-23}$$

By (21) and (22)

$$\kappa[\mathbf{M}^T\mathbf{A}_1 + \mathbf{A}_1\mathbf{M}] = \begin{Vmatrix} a^2 & 0 & (a-b)y \\ 0 & a^2 & (a-b)z \\ (a-b)y & (a-b)z & b^2 \end{Vmatrix}. \qquad (4.2\text{-}24)$$

Since $a \neq b$, it follows from $(15)_6$ and (22) that

$$u = a^2, \quad v = b^2, \quad y = 0, \quad z = 0. \qquad (4.2\text{-}25)$$

EXERCISE 4.2.2

From (23) and (25) show that $\mathbf{M}$ commutes with $\mathbf{M}^T$, and hence by $(15)_{1,3}$ conclude that

$$\mathbf{C}_t^t(s) = e^{-s\mathbf{A}_1}. \qquad (4.2\text{-}26)$$

Case 2. Still on the supposition that $\mathbf{A}_1$ is of the form (21), but regardless of whether (22) does or does not hold, we assume that two monotonous motions can correspond to $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$. Then again, there are tensors $\mathbf{M}$ and $\bar{\mathbf{M}}$ to satisfy (20). Since $\mathbf{M} - \bar{\mathbf{M}}$ is a skew tensor that commutes with $\mathbf{A}_1$ as given by (21), the lemma shows that

$$[\mathbf{M} - \bar{\mathbf{M}}] = \begin{Vmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \qquad (4.2\text{-}27)$$

But also by (15),

$$\mathbf{M}^T\mathbf{A}_2 + \mathbf{A}_2\mathbf{M} = \bar{\mathbf{M}}^T\mathbf{A}_2 + \mathbf{A}_2\bar{\mathbf{M}}, \qquad (4.2\text{-}28)$$

so $\mathbf{M} - \bar{\mathbf{M}}$ commutes with $\mathbf{A}_2$.

EXERCISE 4.2.3

If (22) does not hold, $\mathbf{M} = \bar{\mathbf{M}}$ in case 2. Show that, if $\mathbf{A}_1 = \alpha\mathbf{1}$, then (26) holds.

Accordingly, then, three given tensors $\mathbf{A}_1(t)$, $\mathbf{A}_2(t)$, and $\mathbf{A}_3(t)$ can be the Rivlin-Ericksen tensors corresponding to *at most one* $\mathbf{C}_t^t(s)$ belonging to a monotonous motion. In general, on the contrary, three symmetric tensors taken arbitrarily will fail to be the first three Rivlin-Ericksen tensors of any motion at all, let alone a monotonous motion, since they will fail to satisfy conditions of compatibility expressing the fact that they derive from a velocity field in a region. We shall not take up these conditions since our interest lies in simplifying a constitutive relation when the motion is known to be monotonous.

While Noll's theorem clearly is independent of the dimension of the space, Wang's corollary rests heavily on the use of the three dimensional structure of space.

Noll's theorem (3) suggests an invariant classification of all monotonous motions into three mutually exclusive types:

1. $\mathbf{N}_0^2 = \mathbf{0}$. These motions are called *viscometric flows*.
2. $\mathbf{N}_0^3 = \mathbf{0}$ but $\mathbf{N}_0^2 \neq \mathbf{0}$.
3. $\mathbf{N}_0$ is not nilpotent. In types 1 and 2, since $\operatorname{tr}\mathbf{N}_0 = 0$, the motion is isochoric.

There are interesting examples of all three types, but the simplest, the viscometric flows, are used most in applications. We shall study them further in Chapter 5.

EXERCISE 4.2.4
Show that in types 1 and 2 the motion is isochoric, and also $\operatorname{tr}\mathbf{N}_0^2 = 0$.

EXERCISE 4.2.5
Show that the relative local deformation $\mathbf{F}_t$ of a viscometric flow has the form

$$F_t(\tau) = \mathbf{Q}(\tau)\mathbf{Q}(t)^T[\mathbf{1} + (\tau - t)\kappa\mathbf{Q}(t)\mathbf{N}_0\mathbf{Q}(t)^T],$$

$$\mathbf{N}_0 = \text{const.}, \quad |\mathbf{N}_0| = 1, \quad \mathbf{N}_0^2, \quad \kappa = \text{a scalar field.} \qquad (4.2\text{-}29)$$

Conversely, any relative local deformation of this form corresponds to a viscometric flow.

EXERCISE 4.2.6
Prove that in any monotonous motion

$$\mathbf{A}_2 - \mathbf{A}_1^2 = \kappa^2(\mathbf{N}^T\mathbf{N} - \mathbf{N}\mathbf{N}^T) \qquad (4.2\text{-}30)$$

and hence that

$$\operatorname{tr}\mathbf{A}_1^2 = \operatorname{tr}\mathbf{A}_2 = 2\kappa^2(1 + \operatorname{tr}\mathbf{N}^2). \qquad (4.2\text{-}31)$$

Thus in a viscometric flow

$$\kappa^2 = \frac{1}{2}\operatorname{tr}\mathbf{A}_1^2 = \frac{1}{2}\operatorname{tr}\mathbf{A}_2. \qquad (4.2\text{-}32)$$

EXERCISE 4.2.7 (Noll, Coleman, and Noll)
Prove that the motion whose Cartesian velocity components are

$$\dot{x}_1 = 0, \quad \dot{x}_2 = \kappa x_1, \quad \dot{x}_3 = \lambda x_1 + \mu x_2, \tag{4.2-33}$$

whose λ, μ and κ are constant, is monotonous of type 1 if $\kappa \neq 0$, $\mu = 0$; of type 2 if $\kappa \neq 0$, and that the motion whose Cartesian velocity components are

$$\dot{x}_k = a_k x_k = \text{const.}, \; k = 1, 2, 3, \tag{4.2-34}$$

belongs to Noll's third class if at least one of the a_k does not vanish.

EXERCISE 4.2.8 (Berker, Rajagopal)
Show that

$$\begin{aligned} \dot{x}_1 &= -\Omega(x_2 - g(x_3)), \\ \dot{x}_2 &= \Omega(x_1 - f(x_3)), \\ \dot{x}_3 &= 0, \Omega = \text{const.} \end{aligned} \tag{4.2-35}$$

is a monotonous motion.

For viscometric flows, it follows from its definition that there exists a basis with respect to which

$$[\mathbf{N}] = \begin{Vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{4.2-36}$$

Such a basis is called a viscometric basis.

4.3 Reduction of the Constitutive Relation for a Simple Material in a Monotonous Motion

In view of Wang's corollary, any information that can be determined from $\mathbf{C}_t^t$ in a monotonous motion can be determined also from $\mathbf{A}_1(t)$, $\mathbf{A}_2(t)$, $\mathbf{A}_3(t)$. Therefore, any *functional* of $\mathbf{C}_t^t$ equals, in these motions, a *function* of $\mathbf{A}_1(t)$, $\mathbf{A}_2(t)$, $\mathbf{A}_3(t)$. Consequently the general constitutive relation (3.4-5) may be replaced, as far as motions with constant principal relative stretch histories are concerned, by

$$\mathbf{R}^T\mathbf{T}\mathbf{R} = \mathbf{f}(\mathbf{R}^T\mathbf{A}_1(t)\mathbf{R}, \mathbf{R}^T\mathbf{A}_2(t)\mathbf{R}, \mathbf{R}^T\mathbf{A}_3(t)\mathbf{R}, \mathbf{C}(t)), \tag{4.3-1}$$

$\mathbf{f}$ being a function. A material whose constitutive relation is (1) is called *a material of differential type of complexity* 3. By (1), then, we have the following theorem.

THEOREM (Wang) 4.1. *In the class of monotonous motions a simple material cannot be distinguished from some material of differential type of complexity 3.*

In other words, *no experiment based on interpretation of results for motions with constant principal relative stretch histories can distinguish a general simple material from a differential material of complexity 3.* As we shall see in Chapter 5, the special flows most commonly used to describe the properties of natural fluids are of the kind considered here and hence are of very limited service in exploring the physical properties of those fluids.

An isotropic material of differential type is called a *Rivlin-Ericksen material.* For it, (1) becomes

$$\mathbf{T}(t) = \mathbf{f}(\mathbf{A}_1(t), \mathbf{A}_2(t), \mathbf{A}_3(t), \mathbf{B}(t)), \tag{4.3-2}$$

and when the isotropic material is a fluid,

$$\mathbf{T}(t) = -p(\rho)\mathbf{1} + \mathbf{f}(\mathbf{A}_1(t), \mathbf{A}_2(t), \mathbf{A}_3(t), \rho). \tag{4.3-3}$$

The functions $\mathbf{f}$, in the two cases, are isotropic in the sense that for all symmetric $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{B}$, and for all orthogonal $\mathbf{Q}$

$$\mathbf{f}(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T, \mathbf{Q}\mathbf{A}_2\mathbf{Q}^T, \mathbf{Q}\mathbf{A}_3\mathbf{Q}^T, \mathbf{Q}\mathbf{B}\mathbf{Q}^T \text{ or } \rho) = \mathbf{Q}\mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{B} \text{ or } \rho)\mathbf{Q}^T \tag{4.3-4}$$

Moreover, for a fluid $\mathbf{f}\,(\mathbf{0}, \mathbf{0}, \mathbf{0}, \rho) = \mathbf{0}$.

These reductions may be interpreted in two ways. On the one hand, they enable us to solve easily various special problems concerned with motions having constant principal relative stretch histories. However complicated the response of a material may be in general, in these particular motions we need consider only a simple special constitutive equation. On the other hand, they show that observation of this class of flows is insufficient to tell us much about a material, since most of the complexities of material response are prevented from manifesting themselves.

In Section 6.1 we shall discuss materials of the differential type in somewhat more detail, but in Chapter 5 we shall exploit the present results to obtain specific solutions for viscometric flows of simple fluids.

In a viscometric flow, by definition $\mathbf{N}_0^2 = \mathbf{0}$, and hence

$$\mathbf{A}_3 = \mathbf{A}_4 = \ldots = \mathbf{0}. \tag{4.3-5}$$

Therefore, *in a viscometric flow, a simple fluid cannot be distinguished from some Rivlin-Ericksen fluid of complexity 2.*

For a monotonous motion we see from $(4.2\text{-}15)_1$ that

$$\mathbf{R}^T\mathbf{C}_t^t(s)\mathbf{R} = \exp[-s\kappa(\mathbf{R}^T\mathbf{N}\mathbf{R})^T]\exp[-s\kappa\mathbf{R}^T\mathbf{N}\mathbf{R}]. \tag{4.3-6}$$

Hence any quantity determined by $\mathbf{R}^T\mathbf{C}_t^t\mathbf{R}$ in general is determined here by $\kappa\mathbf{R}^T\mathbf{N}\mathbf{R}$. Referring to the constitutive relation of a simple material, we may set

$$\mathbf{f}(\kappa\mathbf{N}, \mathbf{C}) := \mathcal{R}(\mathbf{R}^T\mathbf{C}_t^t\mathbf{R}, \mathbf{C}) \tag{4.3-7}$$

and so obtain

$$\mathbf{R}^T\mathbf{T}\mathbf{R} = \mathbf{f}(\kappa\mathbf{R}^T\mathbf{N}\mathbf{R}, C) \tag{4.3-8}$$

as an expression for that relation when restricted to monotonous motions. The student will see at once the simpler forms to which (8) reduces for isotropic solids and fluids. For an incompressible fluid the corresponding result is

$$\mathbf{T} = -p\mathbf{1} + \mathbf{f}(\kappa, \mathbf{N}), \mathbf{S} = \mathbf{f}(\kappa, \mathbf{N}), \tag{4.3-9}$$

the function $\mathbf{f}$ being subject to the requirement that

$$\mathbf{f}(\kappa, \mathbf{Q}\mathbf{N}\mathbf{Q}^T) = \mathbf{Q}\mathbf{f}(\kappa, \mathbf{N})\mathbf{Q}^T \tag{4.3-10}$$

for all orthogonal tensors $\mathbf{Q}$ and for all $\mathbf{N}$ such that $|\mathbf{N}| = 0$ and $\mathbf{N}^2 = \mathbf{O}$. From (6) we see that $\mathbf{R}^T\mathbf{C}_t^t\mathbf{R}^T$ is unchanged when κ, $\mathbf{N}$ replaced by $-\kappa$ and $-\mathbf{N}$. Hence

$$\mathbf{f}(-\kappa, -\mathbf{N}) = \mathbf{f}(\kappa, \mathbf{N}). \tag{4.3-11}$$

An important monotonous flow we shall encounter often is the steady lineal flow:

$$\dot{x}_1 = 0, \quad \dot{x}_2 = v(x_1), \quad \dot{x}_3 = 0. \tag{4.3-12}$$

A special instance of this class is the simple shearing flow (2.2-13).

The relations (9) and (10) provide the starting point for the analysis in Chapter 5.

5

Some Flows of Incompressible Fluids in General

5.1 Stress of an Incompressible Fluid in Viscometric Flows

Noll's fundamental theorem on monotonous motions can be stated as

$$\mathbf{F}_0(\tau) = \mathbf{Q}(\tau)e^{t\mathbf{M}_0}, \ \mathbf{Q}(0) = \mathbf{1}, \tag{5.1-1}$$

and so we shall consider it during the first part of this section. $\mathbf{Q}$ is an orthogonal tensor, and $\mathbf{M}_o$ is a constant tensor. In Chapter 4 the theorem was stated as

$$\mathbf{F}_0(t) = \mathbf{Q}(t)e^{t\kappa\mathbf{N}_0}, \ \mathbf{Q}(0) = \mathbf{1}, \ |\mathbf{N}_0| = 1. \tag{4.2-3}$$

Thus

$$\kappa\mathbf{N}_0 = \mathbf{M}_0. \tag{5.1-2}$$

Hence $|\kappa| = |\mathbf{M}_0|$. The quantity $\kappa\mathbf{N}_0$ was defined as $\mathbf{M}$ in (4.2-12). We note that if $\kappa = 0$, then $\mathbf{M}_0 = \mathbf{0}$, but $\mathbf{N}_0$ may be an arbitrary unit tensor.

We shall find it convenient at first to express the determinate part of the stress $\mathbf{S}$ in terms of $\mathbf{M}$. We introduce

$$\mathbf{M}(t) := \mathbf{Q}(t)\mathbf{M}_0\mathbf{Q}^T(t). \tag{5.1-3}$$

The reduced constitutive equation for the determinate part of the stress of an incompressible fluid undergoing a monotonous motion, which has been given by (4.3-9), can now be expressed as

$$\mathbf{S} = \mathbf{J}(\mathbf{M}), \tag{5.1-4}$$

where $\mathbf{J}$ is an isotropic, tensor-valued function:

$$\mathbf{J}(\mathbf{Q}\mathbf{M}\mathbf{Q}^T) = \mathbf{Q}\mathbf{J}(\mathbf{M})\mathbf{Q}^T \tag{5.1-5}$$

for every orthogonal tensor $\mathbf{Q}$. Hence it follows that

$$\mathbf{J}(0) = p\mathbf{1}, \tag{5.1-6}$$

for some constant p.

The requirements of frame indifference and material symmetry impose no further restrictions upon the response $\mathbf{J}$ of the fluid. In general $\mathbf{M}$ will be functions of place alone.

For a viscometric flow we may use (4.2-14), (4.2-16), (2), and (3) to conclude that

$$\mathbf{M}^2 = \mathbf{0}, \ \operatorname{tr} \mathbf{M} = 0. \tag{5.1-7}$$

Accordingly, henceforth the argument $\mathbf{M}$ in (4) and (5) will be restricted by these conditions.

In celebrated work of the late 1940s and early 1950s, Rivlin discovered and interpreted solutions of the equations of motion for some important viscometric flows of certain classes of nonlinear fluids. Here we shall follow later and more general treatments of the problem as a whole by Coleman, Markovitz, and Noll.

First we determine the most general stress compatible with the constitutive equation of an incompressible fluid undergoing a viscometric flow. To that end we shall use a viscometric basis (see 4.2-36), with respect to which

$$[\mathbf{M}] = \kappa \begin{Vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \tag{5.1-8}$$

Looking back at (3), we see that the viscometric basis may change with time and from point to point in space. Generally it is not the natural basis of any coordinate system.

EXERCISE 5.1.1

Use (4) and (5) to show that the components of $\mathbf{S}$ with respect to a viscometric basis are functions of κ alone, $\dot{\kappa} = 0$, and κ is at each instant a function of place in the shape of the fluid body we consider.

While our conclusions follow from general calculations, it helps to visualize them in terms of a simple example. To this end we consider steady, lineal flows, specified as follows in suitable Cartesian coordinates:

$$\dot{\mathrm{x}}_1 = 0, \ \dot{\mathrm{x}}_2 = v(x_1), \ \dot{\mathrm{x}}_3 = 0.$$

The shearing $\kappa = v'(x_1)$. In this example we may speak of the "plane of flow" when in a general argument we mean in reference to (4.2-35) the "plane of i_1 and i_2." Here the Cartesian coordinates provide the viscometric basis.

A reflection across the plane of the flow should be expected to leave the whole stress system invariant. For that reflection

$$[\mathbf{Q}] = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}, \tag{5.1-9}$$

from whence (8) we see that $\mathbf{QNQ}^T = \mathbf{N}$. Therefore, according to (4) and (5), $\mathbf{QSQ}^T = S$. Direct calculation with (9) yields

$$[\mathbf{QSQ}^T] = \begin{Vmatrix} S_{11} & S_{12} & -S_{13} \\ S_{12} & S_{22} & -S_{23} \\ S_{13} & -S_{23} & S_{33} \end{Vmatrix}. \tag{5.1-10}$$

Hence $S_{13} = -S_{13}$ and $S_{23} = -S_{23}$, and

$$S_{13} = S_{23} = 0. \tag{5.1-11}$$

Therefore, there are functions τ, σ_1, and σ_2 of κ alone such that the components of $\mathbf{S}$ satisfy the following relations:

$$\begin{aligned} S_{12} &= T_{12} = \tau(\kappa), \\ S_{11} - S_{33} &= T_{11} - T_{33} = \sigma_1(\kappa), \\ S_{22} - S_{33} &= T_{22} - T_{33} = \sigma_2(\kappa). \end{aligned} \tag{5.1-12}$$

It thus follows that

$$[\mathbf{S}] = \begin{Vmatrix} \sigma_1(\kappa) + S_{33} & \tau(\kappa) & 0 \\ \tau(\kappa) & \sigma_2(\kappa) + S_{33} & 0 \\ 0 & 0 & S_{33} \end{Vmatrix} \tag{5.1-13}$$

relative to the viscometric basis.

It follows from (4), (6), and (8) that

$$\mathbf{S} = p\mathbf{1} \text{ when } \kappa = 0. \tag{5.1-14}$$

Equations (13) and (14) imply that

$$\tau(0) = 0, \ \sigma_1(0) = 0, \ \sigma_2(0) = 0 \tag{5.1-15}$$

and

$$S_{33} = p. \tag{5.1-16}$$

By choosing $\mathbf{Q}$ in (5) to be

$$[\mathbf{Q}] = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{5.1-17}$$

we conclude that

$$\begin{aligned} \tau(-\kappa) &= -\tau(\kappa), \\ \sigma_1(-\kappa) &= \sigma_1(\kappa), \\ \sigma_2(-\kappa) &= \sigma_2(\kappa) \end{aligned} \tag{5.1-18}$$

Thus τ is an odd function of the shearing, while σ_1 and σ_2 are even functions.

In general the functions τ, σ_1, and σ_2 may vary from one fluid point to another. In this book that possibility has been set aside by our decision to consider only homogeneous bodies of fluid.

Next, it follows from (2), (3), (4.2-14), and (8) that

$$\mathbf{M} = \kappa\mathbf{N} \text{ and } [\mathbf{N}] = \begin{Vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \tag{5.1-19}$$

We recall that (4) can be written as

$$\mathbf{S} = \mathbf{f}(\kappa, \mathbf{N}) \tag{4.3-9}$$

and hence (5) takes the form

$$\mathbf{f}(\kappa, \mathbf{Q}\mathbf{N}\mathbf{Q}^T) = \mathbf{Q}f(\kappa, \mathbf{N})\mathbf{Q}^T \tag{4.3-10}$$

for every orthogonal $\mathbf{Q}$. It can also be shown that

$$\mathbf{f}(-\kappa, -\mathbf{N}) = \mathbf{f}(\kappa, \mathbf{N}). \tag{4.3-11}$$

As a consequence of (6),

$$\mathbf{f}(0, \mathbf{N}) = p\mathbf{1}. \tag{5.1-20}$$

Statements (4.3-1), (4.3-11), and (20) are the only restrictions due to frame indifference and material symmetry upon the response $\mathbf{f}$.

We recall that in an incompressible simple fluid

$$\mathbf{T} = -p\mathbf{1} + \mathbf{S}, \tag{3.7-4}$$

in which p is undetermined. Thus, without loss of generality, we may set

$$T_{33} = -p, \tag{5.1-21}$$

and consequently (compare (16))

$$S_{33} = \bar{p} = 0. \tag{5.1-22}$$

Henceforth in this book we shall adopt (22). If we prefer a statement that does not mention a basis, we may write one down by combining (4) with (13), (19), and (22):

$$\mathbf{S} = \mathbf{f}(\kappa, \mathbf{N}) = \tau(\kappa)(\mathbf{N} + \mathbf{N}^T) + \sigma_1(\kappa)\mathbf{N}^T\mathbf{N} + \sigma_2(\kappa)\mathbf{N}\mathbf{N}^T. \tag{5.1-23}$$

In special theories of incompressible materials it is customary to normalize p by the requirement $p = -\frac{1}{3}tr\mathbf{T}$. For example, for the Navier-Stokes theory of fluids this condition is a trivial consequence of incompressibility. In fact, we could have chosen this requirement in place of (22). However, we should caution students that they can not make both choices simultaneously. For the study of viscometric flows we find (22) more convenient.

The following exercise presents a conclusion that was obtained before Noll had constructed his theory of monotonous flows.

EXERCISE 5.1.2 (Criminale, Ericksen, and Filbey)
Show that in any viscometric flows $\mathbf{S}$ may be expressed in terms of the Rivlin-Ericksen tensors (2.2-12) as follows:

$$\mathbf{S} = \frac{\tau(\kappa)}{\kappa}\mathbf{A}_1 + \frac{1}{2}\frac{\sigma_1(\kappa) - \sigma_2(\kappa)}{\kappa^2}\mathbf{A}_2 + \frac{\sigma_2(\kappa)}{\kappa^2}\mathbf{A}_1^2 \tag{5.1-24}$$

EXERCISE 5.1.3
Show that if (23) is satisfied, then so are (20), (4.3-10), and (4.3-11) provided (15) and (18) hold.

The assertion established by doing this exercise shows that when the deformation history corresponds with a viscometric flow, the requirements of frame indifference and material symmetry impose no further restrictions on the response of a fluid.

The relation (23), although differently derived, is a purely algebraic consequence of (4.3-9) and (4.3-10); to establish that fact, it is easier to use a representation[1] for isotropic mappings. The details are left to the student.

The relation (23) was discovered through specializing to viscometric flows the general response of an incompressible fluid. As it stands, however, it defines a Rivlin-Ericksen fluid of complexity 2 (compare 4.3) such as to have arbitrarily given functions τ, σ_1, and σ_2, subject only to the restrictions (15) and (18).

This observation shows that the generality maintained so far is not superfluous. No further restriction of τ, σ_1, and σ_2 can be general in the class of incompressible simple fluids. *For them the theory of viscometric flows in general and that for Rivlin-Ericksen fluids of complexity 2 are equivalent.*

In theories of the mechanics of fluids it is often assumed that the stress power w,

$$\mathrm{w} := \mathbf{T}\cdot\mathbf{G} = \mathbf{T}\cdot\mathbf{D}, \tag{5.1-25}$$

[1] See pp. 215 and 393 of C.-C. Wang, "A new representation theorem for isotropic functions," *Archive for Rational Mechanics and Analysis* 36 (1970): 166–223; 43 (1971): 392–95. A functional basis for isotropic mappings of the space of tensors into the space of symmetric tensors is

$$\mathbf{1}, \mathbf{N}_+, \mathbf{N}_+^2, \mathbf{N}_-^2, \mathbf{N}_-\mathbf{N}_+\mathbf{N}_-, \mathbf{N}_-\mathbf{N}_+ - \mathbf{N}_+\mathbf{N}_-,$$

in which $\mathbf{N}_+$ and $\mathbf{N}_-$ are the symmetric and skew parts of $\mathbf{N}$. When $\mathbf{N}^2 = 0$, this basis and the basis

$$\mathbf{1}, \mathbf{N} + \mathbf{N}^T, \mathbf{N}^T\mathbf{N}, \mathbf{N}\mathbf{N}^T$$

are equivalent.

should not be negative. In the present context that adscitious inequality takes the form

$$\kappa\tau(\kappa) > 0 \text{ if } \kappa \neq 0, \tag{5.1-26}$$

as is plain from (13), $(14)_2$, and $(4.2\text{-}15)_3$. This inequality is invariant under requirement (18). If τ is continuous at $\kappa = 0$, then from (18) it follows that in Exercise 7.2.4 we shall see that for one common kind of fluid the stress power is not of one sign at all times and places.

5.2 Viscometric Functions: Normal-Stress Effects in Steady Viscometric Flows

In Section 5.1 we showed that the stresses in any incompressible fluid in any viscometric flow are given by (5.1-12) with the functions τ, σ_1, and σ_2 restricted by (5.1-15) and (5.1-18).

The functions τ, σ_1, and σ_2 are the *viscometric functions* of the fluid whose constitutive equation reduces in a viscometric flow to (5.1-13) and (5.1-14). They are determined uniquely by the response that defines the fluid in the first place. Conversely, however, it is obvious that infinitely many different fluids share the same three viscometric functions. Therefore, the data from experiments that presume that a viscometric flow takes place are far from sufficient to distinguish one fluid from another in general.

Moreover, while we have deduced the existence of viscometric functions by specializing the constitutive equation (4.3-9) for monotonous motions to viscometric flows, even more general theories when specialized to such flows may lead to the existence of such functions. Viscometric behavior is not limited to simple fluids.

The linear instance of (4.3-12) is the *steady simple shearing* (2.2-13). For it $v(x_1) = \kappa x_{11}$, $\kappa =$ const. *A fortiori*, it may be produced without application of body force in any homogeneous, incompressible simple fluid. Then the formula (5.1-4) relating the stresses may be interpreted immediately. The shear stress that must be supplied on a plane $x_1 =$ const. in order to produce the flow is $\tau(\kappa)$. The normal traction on this plane is T_{11}, and that on the plane of flow is T_{33}. Because the pressure p is indeterminate to within an arbitrary constant, either of these tractions but not both may be given any constant value. If we choose to leave the plane $x_3 =$ const. free, then $T_{33} = 0$, and (5.1-12) yields

$$T_{11} = \sigma_1(\kappa),$$
$$T_{22} = \sigma_2(\kappa). \tag{5.2-1}$$

Thus a fixed normal traction on the plane x_1 = const., determined by κ and by the nature of the fluid, must be supplied; so must a normal traction on the planes x_2 = const., which are normal to the flow. The necessity of these normal tractions provides an example of what are called *normal-stress effects*. In particular, the conclusion (1) shows that in general *shear stress alone is insufficient to produce simple shearing: Suitable and generally unequal normal tractions, determined by the nature of the fluid, must be supplied in order to maintain the flow.*

The functions σ_1 and σ_2 are called the *normal-stress difference functions*, while τ is called the *shear-stress function*. The *shear-viscosity function* μ is defined as follows:

$$\mu(\kappa) := \frac{\tau(\kappa)}{\kappa}, \quad \kappa \neq 0. \tag{5.2-2}$$

The function μ is even and positive except possibly when $\kappa = 0$. In most cases of interest the fluid will be such that $\tau(\kappa) = o(\kappa)$ as $\kappa \to 0$, and $\mu(0)$ can be defined as $\lim_{\kappa \to 0} \mu(\kappa)$. If $\mu(0)$ exists, it is called the *shear-viscosity constant* or *natural shear viscosity* of the fluid.

Loosely, we shall sometimes refer also to the three functions μ, σ_1, and σ_2 as the *viscometric functions*. The reader is expected to bear in mind without further remark that while the function τ exists for any fluid susceptible of undergoing a viscometric flow at all, use of the value $\mu(0)$ requires the further assumption just stated.

5.3 Position of the Classical Theory of Viscometry

The classical theory of viscometry is based on the Navier-Stokes constitutive relation for incompressible fluids:

$$\mathbf{T} = -p\mathbf{1} + 2\mu\mathbf{D}, \quad \operatorname{tr}\mathbf{D} = 0. \tag{5.3-1}$$

EXERCISE 5.3.1
Show that according to the Navier-Stokes theory, the shear-viscosity function is constant and its value is the natural viscosity:

$$\mu(\kappa) \equiv \mu(0) = \text{const.} \tag{5.3-2}$$

while the normal-stress differences vanish identically:

$$\sigma_1(\kappa) \equiv \sigma_2(\kappa) \equiv 0. \tag{5.3-3}$$

Of course it should be obvious that the converse is false: If the viscometric functions satisfy (2) and (3), the constitutive relation (1) does not follow. The formulae (1) and (2) define the *classical* or *Navier-Stokes theory of viscometry*. To accept the Navier-Stokes theory of viscometry does not require us to accept the Navier-Stokes theory of fluids for general flows, since infinitely many other constitutive relations for fluids also lead to the particular viscometric functions (2) and (3).

In general, if we assume that τ and σ_1 and σ_2 have four continuous derivatives at $\kappa = 0$, then by (5.1-18) we see that

$$\begin{aligned} \tau(\kappa) &= \mu_0\kappa + \mu_1\kappa^3 + O(\kappa^5), \\ \sigma_1(\kappa) &= s_1\kappa^2 + O(\kappa^4), \\ \sigma_2(\kappa) &= s_2\kappa^2 + O(\kappa^4), \end{aligned} \tag{5.3-4}$$

where μ_0, μ_1, s_1, and s_2 are constants. Of course $\mu_0 = \mu(0)$. Thus the effects of second order in κ are normal-stress effects, while departure from the classical proportionality of shear stress to shearing is, generally, an effect of third order in κ. Roughly speaking, noticeable departures from the classical behavior as described by (1) may be expected for σ_1 and σ_2 at smaller shearings than for τ. Still more roughly, *normal-stress effects can be expected to manifest themselves within the range in which the response of the shear stress remains classical.*

"Expected" in these remarks refers to the confidence with which experimenters commonly assume that empirical functions are differentiable several times. The theory by itself provides nothing for or against (4). Moreover, even if (4) is valid, there is nothing in our analysis to prove that $s_1 \neq 0$ and $s_2 \neq 0$. There are real fluids that show at low shearings no normal-stress effects at all. Such fluids are said to exhibit "shear thinning" or "shear thickening."

The incompressible, elastic, Eulerian fluid defined by

$$\mathbf{T} = -p\mathbf{1}, \tag{5.3-5}$$

is included formally in the Navier-Stokes theory as the instance in which $\mu = 0$. The Eulerian fluid (5.3-5) is often called "inviscid." In referring to Navier-Stokes fluids, we shall assume without further remark that

$$\mu \equiv \mu_0 = \text{const.} > 0, \tag{5.3-6}$$

trusting that students will not be confused by our occasional mention, in reference to the Euler Theory, of forms that statements about the Navier-Stokes theory take if $\mu = 0$.

5.4 Dynamics of the Main Viscometric Flows

The field of determinate stress corresponding with any viscometric flow of an incompressible simple fluid is given by (5.1-12) or (5.1-13). The viscometric functions τ, σ_1, and σ_2 are determined uniquely by the response of the fluid and hence are the same for all viscometric flows that it may undergo. Both κ and $\mathbf{N}$ are generally functions of place and time. The shearing κ is a scalar, and $\mathbf{N}$ is a tensor such that $|\mathbf{N}| = 1$, $\mathbf{N}^2 = \mathbf{0}$, and hence $\operatorname{tr}\mathbf{N} = 0$. The orthonormal basis with respect to which $\mathbf{N}$ has the special matrix of components (4.2-36) may vary with place and time and does not need be the natural basis of any coordinate system. The scalar p, which equals $-T_{33}$, is not determined by the deformation history. In general, unless a suitably chosen body is supplied, the system of stresses (5.1-13) will fail to satisfy Cauchy's first law (2.7-5), which expresses the balance of linear momentum.

Since we consider only homogeneous fluid bodies, for the compatibility of a flow of a homogeneous incompressible body with any lamellar field of body force $\mathbf{b}$ it suffices to take $\mathbf{b}$ as $\mathbf{0}$, that is, to determine such particular flows as may be effected by applying suitable boundary tractions alone.[2]

One such viscometric flow has already been exhibited, the rectilinear shearing (2.2-13). Generally the velocity field that meets given dynamical requirements depends upon the viscometric functions, as we shall see.

Many of the conclusions we shall now present are due in principle to Rivlin and Ericksen; we shall follow the presentation of Coleman and Noll, who, influenced by the work of Ericksen, were the first to provide a fully general treatment.

EXAMPLE 5.1 Shearing Flow

We shall now find the most general steady linear flow (4.3-12) that can be effected by boundary tractions and lamellar body force in a homogeneous fluid body whose viscometric functions are τ, σ_1, and σ_2. For steady flow, the basis with respect to which $\mathbf{N}$ has the special form (4.2-36) is the natural basis of the coordinate system used, and $\mathbf{N} = \text{const}$. The shearing $\kappa = v'(x_1)$.

We shall employ Cauchy's first law in the form (2.7-5), which we rewrite here as

$$\operatorname{div}\mathbf{S} - \rho \operatorname{grad}\varphi = \rho\ddot{\mathbf{x}}; \tag{5.4-1}$$

$\mathbf{S}$ is the determinate stress, given by $(3.7\text{-}4)_2$, and φ is defined by (3.7-5). For steady shearing flow, $\ddot{\mathbf{x}} \equiv 0$, and in view of (5.1-4) and (4.3-9) we see that $\mathbf{S}$ is of a function of x_1 only. Hence (1) reduces to the following system of differential equations:

$$\begin{aligned} \partial_{x_1} S_{11} - \rho\partial_{x_1}\varphi &= 0, \\ \partial_{x_1} S_{12} - \rho\partial_{x_2}\varphi &= 0, \\ \rho\partial_{x_3}\varphi &= 0. \end{aligned} \tag{5.4-2}$$

[2] See section IV.8 of volume 1 of C. Truesdell, *A First Course in Rational Continuum Mechanics* (New York: Academic Press, 1991) for a detailed discussion of homogeneous motions of a simple body.

The last of these equations shows that φ is independent of x_3. In virtue of the first two, since $\mathbf{S}$ is a function of x_1 only,

$$\partial_{x_2}^2 \varphi = 0, \quad \partial_{x_1}\partial_{x_2}\varphi = 0, \quad \partial_t \partial_{x_2}\varphi = 0. \tag{5.4-3}$$

Hence, φ is an affine function of x_2,

$$\rho\varphi = -ax_2 + k(x_1) + h(t), \tag{5.4-4}$$

a being an assignable constant, which is called the *specific driving force* of the flow, and the functions k and h are arbitrary. Again using the fact that $\mathbf{S}$ is a function of x_1 alone, we substitute (4) into $(2)_2$ and integrate to obtain

$$T_{12} = S_{12} = -ax_1 + c, \quad c = \text{const.} \tag{5.4-5}$$

Thus, no matter what the speed profile v is, the shear stress must be an affine function of x_1. Moreover, the shear stress corresponding to a given specific driving force is the same in all fluids; it is unaffected by the constitutive equation. Substitution of (4) into $(2)_1$ followed by integration yields

$$S_{11} = k(x_1) + b, \quad b = \text{const.} \tag{5.4-6}$$

Conversely, if (4), (5), and (6) hold, the conditions of compatibility (2) are satisfied.

The entire stress system may be calculated as follows. First, by use of (6), (4), and (4.7-5) we show that

$$\begin{aligned} T_{11} &= S_{11} - p = S_{11} - (p + \rho\varpi) + \rho\varpi, \\ &= ax_2 + b + \rho\varpi - h(t), \end{aligned} \tag{5.4-7}$$

in which ϖ, the potential of the body force, is defined by (2.6-3). Because $\partial_{x_2}(T_{11} - \rho\varpi) = a$, the constant a may be interpreted also as the gradient of $T_{11} - \rho\varpi$ in the direction of motion. Second, by use of (7) and (5.1-4) we find that

$$\begin{aligned} T_{22} &= (T_{22} - T_{11}) + T_{11}, \\ &= \sigma_2(\kappa) - \sigma_1(\kappa) + ax_2 + b + \rho\varpi - h(t), \\ T_{33} &= (T_{33} - T_{11}) + T_{11}, \\ &= -\sigma_1(\kappa) + ax_2 + b + \rho\varpi - h(t). \end{aligned} \tag{5.4-8}$$

With any function v and with $\kappa = v'(x_1)$, the normal stresses are delivered by these formulae. Combining (5) and $(5.1\text{-}12)_1$ yields a differential equation to determine the function v:

$$\tau(\kappa) = \tau(v'(x_1)) = -ax_1 + c. \tag{5.4-9}$$

The arbitrary constants a and c are to be assigned, and then $v(x_1)$ is determined by integrating (9). *Thus the speed profile v, which so far has been arbitrary, is determined to within three arbitrary constants by the shear-stress function τ.* This determination results from the balance of linear momentum, on the assumption that only a conservative body force be brought to bear. The most important of the three assignable constants is a, the specific driving force.

EXERCISE 5.4.1
Show that these same conclusions follow by first using the vorticity equation and then determining φ by inspection of (1).

If we take $a = 0$ and assume that τ is invertible, (9) requires that $\kappa = \text{const.}$, and we recover the conclusions already derived for simple shearing. The foregoing analysis proves that if the body force is conservative and τ is invertible, then simple shearing is *the only lineal flow* (4.3-12) *that can be produced when a, the specific driving force, is 0.* While simple shearing is a universal transplacement,[3] matters when $a \neq 0$ are different, for them (9) shows the speed profile v will depend upon the fluid. Thus other lineal shearings, if possible at all, fail of being universal solutions for incompressible fluids. We shall now work out the details in a major instance.

EXAMPLE 5.2 Channel Flow
We seek a solution that represents the flow of a body adhering to stationary infinite plates $x_1 = \pm d,\ d > 0$. Thus we require of the speed profile v that

$$v(d) = v(-d) = 0. \qquad (5.4\text{-}10)$$

We assume the shear-stress function τ to be invertible with inverse, say ζ, which is necessarily an odd function. Then (9) yields

$$\kappa = v'(x_1) = \zeta(-ax_1 + c). \qquad (5.4\text{-}11)$$

We integrate this equation and impose the conditions (10), thus obtaining

$$\int_{-d}^{d} \zeta(-ax + c)dx = 0. \qquad (5.4\text{-}12)$$

By change of the variable of integration,

$$\int_{-d}^{d} \zeta(ax + c)dx = 0. \qquad (5.4\text{-}13)$$

Because ζ is odd, adding (12) to (13) yields

$$\int_{-d}^{d} [\zeta(ax + c) - \zeta(ax - c)]dx = 0. \qquad (5.4\text{-}14)$$

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[3]A motion is called universal for a given class of bodies subject to a body force **b** if it satisfies the equations of motion for all bodies belonging to that class when subject to the body force **b**. See *A First Course in Rational Continuum Mechanics*, vol. 1, pp. 228–236, for a detailed discussion on universal motions and transplacements of isotropic simple bodies.

We now make the further assumption that τ is continuous. Since it is invertible, it is monotone, and so also is ζ. Therefore, if $c \neq 0$, the integrand in (14) is either positive in all of $(-d, d)$ or negative in all of it, so (14) cannot be true. Thus, necessarily, $c = 0$. Consequently, on the assumption that the shear-stress function τ is continuous and invertible, a one speed profile for the given channel is determined by τ:

$$v(x_1) = \int_{x_1}^{d} \zeta(ay)dy. \tag{5.4-15}$$

Since ζ is an odd function, v is an even function.

In contrast with what occurs in steady simple shearing, the profile is generally not at all the same as that predicted by the Navier-Stokes theory. Indeed, if the shear-viscosity function is constant, then $\zeta(y) = (1/\mu)y$, and (15) yields

$$v(x_1) = \frac{1}{\mu}\int_{x_1}^{d} aydy = \frac{a}{2\mu}(d^2 - x_1^2), \tag{5.4-16}$$

the classical parabolic form. Conversely, if $(16)_2$ holds, (15) shows that ζ is a linear function, and the classical linear formula (5.3-1) for the shear viscosity function results.

The *discharge* D, which is the volume of fluid passing through unit depth of channel in unit time, is given by

$$\begin{aligned} D \equiv \int_{-d}^{d} v(x)dx &= 2\int_{0}^{d} dx \int_{x}^{d} \zeta(ay)dy, \\ &= \frac{2}{a^2}\int_{0}^{ad} y\zeta(y)dy. \end{aligned} \tag{5.4-17}$$

Conversely, if the discharge D is known as a function of a and d, (17) yields

$$\tau^{-1}(ad) = \zeta(ad) = \frac{1}{ad^2}\partial_a(a^2 D).$$

Thus if for a given channel we know D in an interval of values of the driving force a, we may determine the shear-stress function τ uniquely in a corresponding interval. In particular, the classical formula

$$D = \frac{2ad^3}{3\mu} \tag{5.4-18}$$

holds if and only if the shear-viscosity function is linear and D is assumed to be a bounded function of a.

Speed profile, discharge, and shear-stress function determine one another and are unaffected by the normal-stress difference functions σ_1 and σ_2. If (18) holds, there is no reason to expect the remaining classical formula (5.3-3) to hold as well. Therefore, the classical viscometric tests, which refer to shear viscosity alone, do not tell much about the fluid being tested. If in a particular case a classical formula such as (18) emerges, this fact not only fails to show that the fluid tested obeys the Navier-Stokes constitutive equation in general but also fails even to establish the Navier-Stokes theory of viscometry. Additional measurements are necessary. In the present case, by (7), the normal tractions on the channel walls $x_1 = \pm d$ do not differ from those predicted by the classical theory. By (8), however, those on the flow planes ($x_3 =$ const.) and those on planes normal to the flow ($x_2 =$ const.) may be entirely different.

Since these normal tractions are difficult to interpret, we turn to a different class of flows, in which normal-stress effects are more striking.

EXAMPLE 5.3 Helical Flows in General

For helical flow described in cylindrical coordinates, namely

$$\dot{r} = 0, \quad \dot{\theta} = \omega(r), \quad \dot{z} = u(r), \tag{5.4-19}$$

each fluid-point remains upon a fixed cylinder $r =$ const., on which its motion describes a helix, whose pitch is the same for all fluid-points on any one cylinder. We set $f := \omega', \quad h := u'$. Then

$$\kappa^2 = r^2 f(r)^2 + h(r)^2.$$

Let $\{\mathbf{e}_k(x)\}$, k = 1,2,3, be an orthonormal basis tangent to the coordinate curves at $\mathbf{x}$, and

$$\mathbf{i}_1 := \mathbf{e}_1, \ \mathbf{i}_2 := \alpha\mathbf{e}_2 + \beta\mathbf{e}_3, \ \mathbf{i}_3 := -\beta\mathbf{e}_2 + \alpha\mathbf{e}_3,$$

where the functions α and β are defined as follows:

$$\alpha := \frac{r}{\kappa} f(r), \quad \beta := \frac{1}{\kappa} h(r), \ \alpha^2 + \beta^2 = 1. \tag{5.4-20}$$

The conclusions (5.1-12) and (5.1-16) apply to the components of $\mathbf{T}$ relative to the basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$:

$$\begin{aligned} &T_{12} = \tau(\kappa), \qquad T_{13} = 0, \qquad T_{23} = 0, \\ &T_{11} - T_{33} = \sigma_1(\kappa), \\ &T_{22} - T_{33} = \sigma_2(\kappa). \end{aligned} \tag{5.4-21}$$

The physical components of $\mathbf{T}$ in cylindrical coordinates are its components with respect to the orthonormal basis $\mathbf{e}_i$. Denoting these components by $T_{rr}, T_{r\theta}$, etc., we find that

$$T_{r\theta} = \mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_2 = \mathbf{i}_1 \cdot (\alpha\mathbf{T}\mathbf{i}_2 - \beta\mathbf{T}\mathbf{i}_3)$$

$$= \alpha T_{12} - \beta \mathrm{T}_{13},$$
$$T_{\theta z} = \mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_3 = (\alpha \mathbf{i}_2 - \beta \mathbf{i}_3) \cdot (\beta \mathbf{T}\mathbf{i}_2 + \alpha \mathbf{T}\mathbf{i}_3),$$
$$= \alpha\beta(T_{22} - T_{33}) + (\alpha^2 - \beta^2) T_{23},$$

and so on. From these statements and (21) we calculate the stress system in terms of the viscometric functions:

$$\begin{aligned} T_{r\theta} &= \alpha \tau(\kappa), \\ T_{rz} &= \beta \tau(\kappa), \\ T_{\theta z} &= \alpha\beta\sigma_2(\kappa), \\ T_{rr} - T_{zz} &= \sigma_1(\kappa) - \beta^2 \sigma_2(\kappa), \\ T_{\theta\theta} - T_{zz} &= (\alpha^2 - \beta^2)\sigma_2(\kappa). \end{aligned} \tag{5.4-22}$$

It remains now to see whether the functions f and h can be chosen in such a way as to make these stresses compatible with Cauchy's first law of motion when the body force is conservative.

EXERCISE 5.4.2

Since the physical components of the determinate stress **S** are functions of r and θ, show that, Cauchy's first law as expressed by (1) assumes the form

$$\begin{aligned} \partial_r S_{rr} + \frac{1}{r}(S_{rr} - S_{\theta\theta}) - \rho \partial_r \varphi &= \rho r \varpi^2, \\ r \partial_r S_{r\theta} + 2 S_{r\theta} - \rho \partial_\theta \varphi &= 0, \\ \partial_r S_{rz} + \frac{1}{r} S_{rz} - \rho \partial_z \varphi &= 0. \end{aligned} \tag{5.4-23}$$

Hence

$$\begin{aligned} \partial_r (r^2 T_{r\theta}) &= -rd, \\ \partial_r (r T_{rz}) &= -ra, \\ T_{rr} &= \rho \varpi + k(r,t) + az + d\theta, \\ \partial_r k(r,t) + \tfrac{1}{r}(T_{rr} - T_{\theta\theta}) &= -\rho r \varpi^2, \end{aligned} \tag{5.4-24}$$

where a and d are arbitrary constants.

Integration of the first two of these equations yields

$$T_{r\theta} = \left(\frac{c}{r^2} - \frac{d}{2}\right),$$

$$T_{rz} = \left(\frac{b}{r} - \frac{ra}{2}\right), \tag{5.4-25}$$

where b and c are arbitrary constants. From $(22)_{1,2}$ we see that

$$\alpha = \frac{T_{r\theta}}{\tau(\kappa)}, \beta = \frac{T_{rz}}{\tau(\kappa)} \tag{5.4-26}$$

so by $(20)_3$ and (25),

$$\tau(\kappa) = \gamma := \sqrt{\left(\frac{c}{r^2} - \frac{d}{2}\right)^2 + \left(\frac{b}{r} - \frac{ra}{2}\right)^2}. \tag{5.4-27}$$

At this point we have chosen a positive value for κ, so (5.1-26) has made us take the positive square root for τ. Thus

$$\alpha = \frac{1}{\gamma}\left(\frac{c}{r^2} - \frac{d}{2}\right), \beta = \frac{1}{\gamma}\left(\frac{b}{r} - \frac{ra}{2}\right). \tag{5.4-28}$$

Finally, from (20) and the definition of f and h,

$$\omega' = f = \frac{\kappa\alpha}{r} = \frac{\kappa}{\gamma r}\left(\frac{c}{r^2} - \frac{d}{2}\right),$$
$$u' = h = \kappa\beta = \frac{\kappa}{\gamma}\left(\frac{b}{r} - \frac{ra}{2}\right). \tag{5.4-29}$$

When the four constants a, b, c, d are fixed, γ becomes a known function of r by (27). We shall see later how these constants are determined in specific flows such as that between rotating cylinders and that in a circular pipe. In view of $\kappa^2 = r^2 f^2(r) + h^2(r)$, the conclusion (29) becomes a system of functional equations for determining f and h.

The functional system is easy to solve if the shear-stress function τ is invertible, say,

$$\kappa = \zeta(\tau). \tag{5.4-30}$$

Then

$$\omega' = \frac{\zeta(\tau)}{\tau r}\left(\frac{c}{r^2} - \frac{d}{2}\right),$$
$$u' = \frac{\zeta(\tau)}{\tau}\left(\frac{b}{r} - \frac{ra}{2}\right), \tag{5.4-31}$$

so the two functions w and u occurring in the definition of helical flow are determined to within six arbitrary constants. Conversely, if w and u satisfy (31), the helical flow may be produced by the aid of suitable boundary tractions in the fluid whose shear-stress function is τ.

EXERCISE 5.4.3
Show that from (22), (24), and (30) it follows that

$$T_{rr} - T_{zz} = \sigma_1(r) - \beta^2\sigma_2(r),$$
$$T_{\theta\theta} - T_{zz} = (\alpha^2 - \beta^2)\sigma_2(r),$$
$$T_{rr} = \rho\varpi + \int \left\{ \frac{1}{r}[\alpha^2\sigma_2(r) - \sigma_1(r)] - \rho r \varpi(r)^2 \right\} dr + az + d\theta + g(t), \tag{5.4-32}$$

where the functions σ_Γ are defined as follows:

$$\sigma_\Gamma(r) := \sigma_\Gamma[\zeta(\tau(r))], \ \Gamma = 1, 2, \tag{5.4-33}$$

the functions $\zeta(\tau)$ and $\tau(r)$ being given by (30) and $(27)_2$, respectively. We shall now interpret these conclusions in two major special cases.

EXAMPLE 3A Flow between Rotating Cylinders
Because $T_{rz} = 0$ in this flow, we set $a := b$, $b := 0$ in $(27)_2$, then $\beta = 0$, and we may take $u = 0$ in (19). In this flow, a simple vortex, the fluid points move in concentric circles with angular speeds $\omega(r)$ determined from $(29)_1$. For the radial stress T_{rr} to be determinate, it is necessary by $(32)_3$ that $d = 0$. By (27), then,

$$\tau(r) = \frac{c}{r^2}. \tag{5.4-34}$$

The one remaining arbitrary constant c is easily interpreted, since the torque F with respect to a point on the axis, applied to unit height of the cylinder $r =$ const., is given by $F = (2\pi r)r(T_{r\theta})$, which by $(25)_1$ is $2\pi c$. That is $c = F/(2\pi)$. This torque F is to be so adjusted that the cylinders $r = R_1$ and $r = R_2$ move with prescribed angular speeds Ω_1 and Ω_2:

$$\omega(R_1) = \Omega_1, \ \Omega(R_2) = \Omega_2. \tag{5.4-35}$$

By $(29)_1$, if τ is invertible,

$$\Omega_2 - \Omega_1 = \int_{R_1}^{R_2} \frac{1}{r}\zeta\left(\frac{F}{2\pi r^2}\right) dr. \tag{5.4-36}$$

A vortex of this kind approximates the flow within a common type of viscometer named after Couette, in which the torque applied to one cylinder is measured as a function of the difference of angular speeds. The corresponding relation from the theory is given by (36). If (36) can be inverted, the inverse ζ of the shear-viscosity function τ is determined as a function of F for given $\Omega_2 - \Omega_1$.

While according to the theory of perturbations within the Navier-Stokes theory the surfaces $z =$ const. sustain an almost uniform pressure when $\varpi = 0$ and $\rho\omega^2$ is negligibly small, we see from (32) that in a general fluid the normal traction

T_{zz} is a function of r. If we fail to supply this traction, as for example on the top surface of the fluid in a Couette viscometer, that surface cannot remain plane. To determine the shape of the free surface, or even to decide whether the fluid will tend to rise or to fall at one or the other cylindrical boundary, is a different matter. It is not obvious that the effect is governed by the viscometric functions alone, and even for the linearly viscous fluid the precise solution is not yet known.

EXAMPLE 3B Flow in a Circular Pipe

For a second instance of helical flow, we now consider a flow straight down a cylindrical pipe of infinite length, and we assume that fluid adheres to the wall. On scant historical basis a flow of this kind is often called a Poiseuille flow. In (27) we take $c := d, d := 0$, so $a = 0$; to keep the velocity gradient finite at $r = 0$, we take also $b = 0$. Then by (27)

$$\tau(r) = \frac{1}{2}ra, \tag{5.4-37}$$

where a is the specific driving force. From (29), if τ is invertible, we obtain the speed profile

$$u(r) = \int_r^R \zeta\left(\frac{1}{2}ay\right) dy. \tag{5.4-38}$$

where R is the radius of the tube. If $\zeta(\tau) = \tau/\mu$, as is true for the Navier-Stokes theory, the speed profile is a parabola,

$$u(r) = \frac{a}{4\mu}(R^2 - r^2), \tag{5.4-39}$$

and conversely, if for given R the speed profile is parabolic for all a, the shear-stress function is linear. For a general fluid the discharge D is given by

$$\begin{aligned} D(a, R) &= 2\pi \int_0^R r\,dr \int_r^R \zeta\left(\frac{1}{2}ay\right) dy \\ &= \pi \int_0^R r^2\zeta\left(\frac{1}{2}ar\right) dr. \end{aligned} \tag{5.4-40}$$

If ζ is linear, the discharge becomes

$$D(a, R) = \frac{\pi a R^4}{8\mu}. \tag{5.4-41}$$

Here we see the famous Hagen-Poiseuille formula or law of the fourth power; we can show easily that, conversely, this law suffices for the shear-stress function to be linear. If (41) holds, we have no assurance that even the Navier-Stokes theory of viscometry is justified, since the nature of the normal-stress functions has no effect on the discharge and hence cannot be determined from its properties.

Hagen and Poiseuille obtained (41) independently in 1839 and 1840, respectively, on the basis of their experiments. In his discussion Hagen assumed that the speed profile was triangular. Stokes, in researches leading to his great memoir of 1845 on the theory of viscous fluids, derived the correct results (39) and (41), but in the published text he omitted (41) and included the parabolic profile only, omitting the discharge of long straight circular pipes and rectangular channels, and compared the resulting formulae with some of the experiments of Bossut and Dubuat. However, the formulae did not at all agree with experiment. Apparently Stokes knew as little of the work of Hagen and Poiseuille as they knew of each other's; today we know that the experiments of Bossut and Dubuat concerned turbulent flows. Here a remark of Dirac deserves to be subjoined:

If there is not complete agreement between the results of one's [theoretical] work and experiment, one should not allow oneself to be too discouraged, because the discrepancy may well be due to minor features that are not properly taken into account and that will get cleared up with further development of the theory.

By specializing $(32)_1$ and (33) to the flow under consideration, again if τ is invertible, we obtain the following expression for the most important difference of normal tractions:

$$T_{rr} - T_{zz} = \sigma_1\left(\zeta\left(\frac{1}{2}ra\right)\right) - \sigma_2\left(\zeta\left(\frac{1}{2}ra\right)\right). \tag{5.4-42}$$

The fact that this difference usually fails to vanish suggests that a column of fluid emerging after flowing through a long pipe will tend to swell or shrink in diameter.

Various relations between the viscometric functions have been put forward as sufficient that the fluid shall swell upon emergence, but they rest upon hypotheses concerning the flow at the exit, and all these hypotheses have been criticized.

There are further interesting special cases of helical flow.

EXAMPLE 5.4 Some Other Steady Viscometric Flows

Torsional flow, given in cylindrical coordinates by the following contravariant components of the velocity field,

$$\dot{r} = 0, \ \dot{\theta} = \omega(z), \ \dot{z} = 0, \tag{5.4-43}$$

and the flow effected between a cone and a plate in spherical polar coordinates by the velocity field

$$\dot{r} = 0, \ \dot{\theta} = 0, \ \dot{\varphi} = \omega(\theta) \tag{5.4-44}$$

have both been studied. Because these flows are viscometric, the stress system needed to effect one is easily expressed in terms of the viscometric functions. That notwithstanding, none of these can satisfy the dynamical equations exactly unless body forces are supplied that are not conservative. To make them agree roughly with Cauchy's First Law when $\mathbf{b} = \mathbf{0}$, it is customary to suppose the accelerations negligible; even so, for a flow between a cone and a plate it is further necessary to suppose θ limited to a very small interval, about $\theta = \frac{1}{2}\pi$.

In any steady viscometric flow, the tensor **S** is completely determined by κ, **N**, and the three viscometric functions τ, σ_1, and σ_2. Thus all phenomena in the entire class of viscometric flows are simply related to one another. The only problem comes in adjusting κ and **N** to make the flow dynamically possible when the body force is lamellar. In this section we have considered the classes of steady viscometric flows for which this adjustment can be made exactly, and we have listed two others in which it is known that an approximate solution is possible. In all these cases, **N** and κ depend upon the nature of the function τ but are not affected by σ_1 and σ_2.

5.5 Some Unsteady Flows

Coleman and Noll studied the *unsteady lineal flows*:

$$\dot{x}_1 = 0, \ \dot{x}_2 = v(x_1, t), \ \dot{x}_3 = 0. \tag{5.5-1}$$

They represent a stack of parallel plane sheets sheared at a rate that depends upon time. Clearly $\mathcal{W} = 1$ always and everywhere.

EXERCISE 5.5.1 (Coleman and Noll)
Leaving the argument x_1 unwritten, letting a prime denote differentiation with respect to x_1, and regarding t as a parameter, show that from (1) we obtain

$$\begin{aligned} \mathbf{F}_t^t(s) &= e^{g(s)\mathbf{N}}, \\ g(s) &:= \kappa(x_1, t - s) - \kappa(x_1, t) \\ \kappa(t) &:= \int_0^t v'(\alpha) d\alpha, \end{aligned} \tag{5.5-2}$$

and [N] has the form (4.2-36) with respect to the coordinate basis. Thus these flows are not monotonous unless they are steady.

From (2) and by use of (4.1-2) we conclude that

$$\mathbf{T} = \mathcal{R}(e^{g(s)\mathbf{N}^T} e^{g(s)\mathbf{N}}) := \mathbf{f}(g(s), \mathbf{N})). \tag{5.5-3}$$

Arguments that parallel step by step those applied to (5.1-12) in 5.1 then show that

$$\begin{aligned} T_{12} &= \tau(g(s)) = -\tau(-g(s)), \\ T_{11} - T_{33} &= \sigma_1(g(s)) = \sigma_1(-g(s)), \\ T_{22} - T_{33} &= \sigma_2(g(s)) = \sigma_2(-g(s)), \end{aligned} \tag{5.5-4}$$

and hence

$$\mathbf{T} = (T_{33})\mathbf{1} + \tau(g(s))(\mathbf{N} + \mathbf{N}^T) + \sigma_1(g(s))\mathbf{N}^T\mathbf{N} + \sigma_2(g(s))\mathbf{N}\mathbf{N}^T. \tag{5.5-5}$$

EXERCISE 5.5.2 (Coleman and Noll)
To make the speed periodic with least period 2θ, suppose that

$$v(x_1, t + \theta) = -v(x_1, t) \tag{5.5-6}$$

for all x_1. Use (2) to show that

$$\kappa(x_1, t + \theta) = -\kappa(x_1, t) + \int_0^\theta \partial_{x_1} v(x_1, \sigma) d\sigma, \tag{5.5-7}$$

so κ is periodic with least period 2θ. Appeal to $(2)_2$ and (4) then shows that the shear stress oscillates with the same frequency as does the velocity, but the normal-stress differences oscillate with double that frequency.

A steady rotation superposed upon a harmonic oscillation along its axis with amplitude depending upon the distance along that axis is specified as follows in Cartesian coordinates:

$$\begin{aligned} \dot{x}_1 &= -\Omega x_2 - F(x_3)\sin(\Omega t), \\ \dot{x}_2 &= \Omega x_1 + F(x_3)\cos(\Omega t), \\ \dot{x}_3 &= 0, \ \Omega = \text{const.} \neq 0. \end{aligned} \tag{5.5-8}$$

EXERCISE 5.5.3 (Rajagopal)
Show that the flow (8) is viscometric, and its viscometric basis oscillates:

$$\kappa = F'(x_3), \ [\mathbf{N}] = \begin{Vmatrix} 0 & 0 & -\sin(\Omega t) \\ 0 & 0 & \cos(\Omega t) \\ 0 & 0 & 0 \end{Vmatrix}; \tag{5.5-9}$$

also $\mathcal{W}$ is independent of t, and $\mathcal{W}^2 = 1 + 4\Omega^2/F^2 > 1$.

Neither of these classes of unsteady flows is dynamically possible except for particular fluids or particular body forces.

5.6 Steady Flow between Rotating Parallel Plates

In 4.2 and, we studied the isochoric flow

$$\dot{x}_1 = -\Omega(x_2 - g(x_3)),\ \dot{x}_2 = \Omega(x_1 - f(x_3)),\ \dot{x}_3 = 0,\ \Omega = \text{const.}, \tag{4.2-35}$$

which represents a body confined between parallel plates rotating at constant and equal angular speeds Ω about axes parallel to the x_3-axis. We have seen that while it is a monotonous flow of Noll's type 3, $\mathbf{A}_1$, $\mathbf{A}_2$, and ϖ still determine $\mathbf{A}_3, \mathbf{A}_4, \ldots$, and thus it has much in common with a viscometric flow. In particular, for an incompressible fluid undergoing this flow the relation (4.1-6) delivering the determinate stress reduces to

$$\mathbf{S} = \mathcal{R}(\mathbf{C}_t^t) =: \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2; \Omega). \tag{5.6-1}$$

Moreover, when Ω is assigned, $\mathbf{A}_1$ and $\mathbf{A}_2$ are determined by f' and g', so we may write

$$\mathbf{S} = \mathbf{h}(f', g'). \tag{5.6-2}$$

In Chapter 7 the student's will see that the condition of adherence to solid boundaries does not generally suffice to set a definite boundary-value problem for a fluid that obeys a nonlinear constitutive equation. Here, on the contrary, from (2) we remark that the dynamical equation must reduce to a system of ordinary differential equations of second order for f and g, and hence the condition of adherence may suffice to determine a solution, which in fact it does, as we shall see in Exercise 6.1, when the fluid is of a special kind.

We return to the analysis. The components of div $\mathbf{S}$ are known functions of f', g', f'', g'', and consequently they are also known functions of x_3 alone, say,

$$\begin{aligned} S_{13,3} &= h_1(x_3),\\ S_{23,3} &= h_2(x_3),\\ S_{33,3} &= h_3(x_3). \end{aligned} \tag{5.6-3}$$

EXERCISE 5.6.1 (Rajagopal)
Show that the acceleration field that corresponds with (4.2-35) is

$$\ddot{x}_1 = -\Omega^2(x_1 - f(x_3)),\ \ddot{x}_2 = -\Omega^2(x_2 - g(x_3)),\ \ddot{x}_3 = 0, \Omega = \text{const.} \neq 0. \tag{5.6-4}$$

The vorticity equation (3.8-5) then yields

$$h_1' = \rho\Omega^2 f',\ h_2' = \rho\Omega^2 g'. \tag{5.6-5}$$

Hence, by integration,

$$h_1 = \rho\Omega^2 f + A, \; h_2 = \rho\Omega^2 g + B, \tag{5.6-6}$$

where A and B are arbitrary functions of t alone.

EXERCISE 5.6.2 (Rajagopal)
Show that

$$\rho\varphi = \frac{1}{2}\rho\Omega^2(x_1^2 + x_2^2) + Ax_1 + Bx_2 + \rho h(x_3), \tag{5.6-7}$$

in which

$$\rho h(x_3) := \int_0^{x_3} h_3(s)ds. \tag{5.6-8}$$

From (6), (3), and (2) we conclude that

$$\begin{aligned} S_{13,3}(f', g', f''g'') &= \rho\Omega^2 \mathrm{f} + A, \\ S_{23,3}(f', g', f'', g'') &= \rho\Omega^2 g + B. \end{aligned} \tag{5.6-9}$$

As the functions on the left-hand sides are determined by (2), which is made specific by the determinate response as reduced to (2), we may regard (9) as a system, generally nonlinear, of second order that the functions f and g must satisfy for given Ω; if such f and g may be found, they select from (4.2-35) such special instances as are compatible with the constitutive equation of the fluid being considered.

EXERCISE 5.6.3
Let $(a, 0, d)$ and $(-a, 0, -d)$ be the coordinates of the centers of rotation of the top and bottom plates, respectively. Show that the fluid adheres to the plates if and only if

$$f(d) = a, \; f(-d) = -a, \; g(d) = g(-d) = 0. \tag{5.6-10}$$

Whether there are solutions f, g of (9) that satisfy (10) will depend upon the prescribed functions $S_{13,3}$ and $S_{23,3}$, that is, upon g in (2), and hence because of (1) upon the determinate response $\mathcal{R}$ of the fluid.

EXERCISE 5.6.4
If

$$f(x_3) = \frac{d}{a}x_3, \quad g(x_3) := 0, \tag{5.6-11}$$

then show that (9) is satisfied if inertia is neglected.

The solution (11) presumably approximate, makes the locus of centers of rotation be the straight line $x_1 = (d/a)x_3$. In Section 9 of Chapter 6 we shall see that for a particular nonlinear fluid such is far from being true of the genuine solution. Also in that Section 9 of Chapter 6 we shall learn that for the Navier-Stokes fluid and for another particular fluid there is no need to neglect inertia or to resort to perturbations in order to find a unique solution corresponding to (10). Perhaps those two fluids are typical in regard to solutions for this class of flows.

5.7 Impossibility of Steady Rectilinear Flow in Pipes

In Section 2.2 the most general form of the stress possible in a fluid undergoing viscometric flow was determined, and in Section 5.4 certain classes of steady viscometric flows were shown to be dynamically possible without bringing to bear body force not conservative. In these classes the streamlines are the same as for a Navier-Stokes fluid in the same circumstances (see Section 5.3), but the distribution of speeds upon them, which is determined by the shear-stress function τ of the fluid, is different.

Up to the present, few fixed or simply moving boundaries are known to correspond to flows for which the Navier-Stokes equations can be solved easily. Nearly all of these give rise to viscometric flows. Those for which the analysis is easy are exhausted by the cases developed, at least in outline, in Section 5.4 where we emphasized the Navier-Stokes theory only to contrast its special features with those manifested by general fluids in the same circumstances. The class next easiest to analyze is defined by flow in an infinitely long tube of constant cross-section which is a bounded, open, simply connected region of the plane. The customary procedure begins by assuming the motion to be a *steady shearing*, namely an acceleration less lineal flow in the direction of a unit vector **k** normal to the cross-section

$$\dot{\mathbf{x}} = v(\mathbf{p})\mathbf{k} \text{ and } v(\mathbf{p}) = 0 \text{ if } \mathbf{p} \in \partial D, \tag{5.7-1}$$

where **p** is the position of a point in $D \cup \partial D$. This flow is isochoric. The fluid points are assumed to move at constant speed down the lines parallel to the walls of the pipe. As we shall see shortly, the assumption $(1)_1$ is easily shown to be compatible with the Navier-Stokes equations provided that v satisfies a certain partial differential equation that is proved to have a unique solution corresponding to the boundary condition $(1)_2$. Thus a unique rectilinear solution exists. Whether it is the only solution of the boundary-value problem of flow in a pipe is a more difficult matter, according to the Navier-Stokes theory rarely asked and today unsettled.

We shall now approach the problem in the same spirit for an arbitrary incompressible fluid. We shall show that, in general, no steady lineal flow exists. For proof it suffices to refer to the analysis carried out for a particular fluid by Ericksen, who discovered this remarkable fact, for a single counterexample disproves a general assertion; nonetheless, use of the apparatus set up in the preceding sections makes it easier to see just why no rectilinear flow can be expected in general and to characterize those special instances in which such a flow is possible.

EXERCISE 5.7.1
Show that the relative description (1.7-2) of the motion corresponding to the velocity field $(1)_1$ is

$$\boldsymbol{\xi} = \mathbf{x} + (\tau - t)v(\mathbf{p})\mathbf{k}. \tag{5.7-2}$$

Substituting (2) into (1.7-6) yields

$$\mathbf{F}_t(\tau) = \mathbf{1} + (\tau - t)\mathbf{M}, \ \ \mathbf{M} = \mathbf{k} \otimes \nabla v, \tag{5.7-3}$$

in which expression ∇ denotes the gradient operator in the plane containing $\mathbf{p}$. Because of (4.2-29), we know that the assumed motions $(1)_1$ are viscometric and that

$$\kappa^2 = |\nabla v|^2.$$

Substitution of (3) into (5.1-13) yields the most general stress field compatible with the constitutive relation of an incompressible fluid in a flow of this kind,

$$\mathbf{S} = +\mu(\mathbf{k} \otimes \nabla v + \nabla v \otimes \mathbf{k}) + \frac{\sigma_1}{\kappa^2}\nabla v \otimes \nabla v + \sigma_2 \mathbf{k} \otimes \mathbf{k}, \tag{5.7-4}$$

where μ is the shear-viscosity function (5.2-2).

EXERCISE 5.7.2 (Noll)
Show that

$$\operatorname{div}\mathbf{S} = \operatorname{div}(\mu\nabla v)\mathbf{k} + \operatorname{div}\left(\frac{\sigma_1}{\kappa^2}\nabla v\right)\nabla v + \frac{\sigma_1}{\kappa}\nabla\kappa. \tag{5.7-5}$$

If

$$h := \int \frac{\sigma_1(\kappa)}{\kappa}\mathrm{d}\kappa - \rho\varphi, \tag{5.7-6}$$

then

$$\text{grad } h = \frac{\sigma_1(\kappa)}{\kappa}\nabla\kappa - \rho \text{ grad } \varphi, \tag{5.7-7}$$

so Cauchy's first law as expressed by (5.4-1) assumes the form

$$\text{div}(\mu\nabla v)\text{k} + \text{div}\left(\frac{\sigma_1}{\kappa^2}\nabla v\right) + \text{grad } h = 0. \tag{5.7-8}$$

Letting z be a coordinate of length along the pipe, we see from (8) that grad h is independent of z. Therefore

$$h = az + g(\mathbf{p}), \quad a = \text{const.}, \tag{5.7-9}$$

and (8) splits into the following two equations:

$$\text{div}(\mu(\kappa)\nabla v) = -a \tag{5.7-10}$$

and

$$\text{div}\left(\frac{\sigma_1(\kappa)}{\kappa^2}\nabla v\right)\nabla v + \nabla g = 0. \tag{5.7-11}$$

The second equation states that g is constant along the *equivels* $v =$ const. That is, $g(\mathbf{p}) = f(v(\mathbf{p}))$, and (11) becomes

$$\text{div}\left(\frac{\sigma_1(\kappa)}{\kappa^2}\nabla v\right) = -f'(v). \tag{5.7-12}$$

For any given fluid the two viscometric functions τ and σ_1 are determined by the response and hence are regarded as given. Accordingly, we have derived two nonlinear partial differential equations, (10) and (12), to be satisfied by the one function v. In our attempt to find a single function v subject to this double requirement, we may choose the function f at will.

In particular cases, the suitable choice of f does indeed render the two equations for v compatible. For example, if the fluid is such that

$$\sigma_1(\kappa) = c\kappa^2\mu(\kappa), \tag{5.7-13}$$

where c is some constant, then the choice $f'(v) = ca$ renders (12) and (10) identical. The Navier-Stokes theory requires that $c = 0$; so do all constitutive relations in which $\sigma_1{:=}0$. In the Navier-Stokes theory, (10) becomes $\mu\Delta v = -a$, in which μ is the shear viscosity and a is the specific driving force, both being assigned constants. This elliptic partial differential equation has a unique solution satisfying the boundary condition $(1)_2$. Works on the Navier-Stokes theory take up in detail the properties of the solutions for various cross-sections a, but we shall not go further into the matter here except to remark upon an important if very easy special case.

For the elliptical cross-section defined by the Cartesian equation

$$\frac{x_1^2}{c^2} + \frac{x_2^2}{b^2} = 1, \quad c > b, \tag{5.7-14}$$

the Navier-Stokes solution of the problem is given by

$$v = -\frac{ac^2b^2}{2\mu(c^2+b^2)}\left(\frac{x_1^2}{c^2}+\frac{x_2^2}{b^2}-1\right), \tag{5.7-15}$$

where we assume, of course, that $\mu > 0$. That (15) is a solution may be verified by substitution, and the uniqueness theorem for the Dirichlet problem for elliptic linear differential equations with constant coefficients assures us that it is the only solution satisfying the boundary condition.

The example of the Navier-Stokes theory shows that the two conditions (10) and (12) may be compatible. If they are, the constant a has a simple interpretation.

EXERCISE 5.7.3

If (10) and (12) are compatible, then show that

$$T_{zz} - \rho\varpi = za + \sigma_2(\kappa) - \int \frac{\sigma_1(\kappa)}{\kappa} d\kappa + g(p). \tag{5.7-16}$$

Hence

$$\partial_z(T_{zz} - \rho\varpi) = a. \tag{5.7-17}$$

Thus a is the specific driving force.

More generally we expect, though it has not been proved, that (10) by itself, with an assigned shear-viscosity function μ, should again be sufficient to determine a unique solution v satisfying the boundary condition. If that is so, then such a v will generally fail to satisfy (12). Again there are exceptions. If the curves $v =$ const. are concentric circles or parallel straight lines, then κ is a function of v, so (12) is always satisfied. Ericksen[4] proved that if μ was analytic and if (13) did not hold, there were always solutions of (10) for which (12) failed to hold, and he conjectured that in fluids for which (13) did not hold, the only common solutions of (10) and (12) if $a \neq 0$ were those obtained here, namely those in which the curves $v =$ const. are parallel straight lines or concentric circles. Fosdick and Serrin[5] provided a complete analysis of the problem. By counterexamples, they show that the conjecture is false, though the main consequences Ericksen deduced from it are true. Their assumptions are as follows:

1. μ is of class C^2 for all κ.

[4]J.L. Ericksen, "Overdetermination of the speed in rectilinear motion of non-Newtonian fluids," *Quarterly of Applied Mathematics* 14 (1956): 318–21. Reprinted in *Rational Mechanics of Materials*, edited by C. Truesdell, International Science Review Series (New York: Gordon and Breach, 1965).

[5]R.L. Fosdick and J. Serrin, "Rectilinear steady flow of simple fluids," *Proceedings of the Royal Society* (London) A333 (1973): 311–33.

2. μ and σ_1/κ^2 are of class C^3 near $\kappa = 0$, and their first and third derivatives vanish at $\kappa = 0$.

3. $\mu\kappa$ is an increasing function of κ for all κ.

Fosdick and Serrin then assert that if (13) does not hold, the only bounded and connected open sets D on which there is a common solution of (10) and (12) that vanishes on ∂D are those for which ∂D is a circle or a pair of concentric circles. They present a proof under the stronger assumption that μ and σ_1/κ^2 are analytic functions of κ^2 near $\kappa = 0$. They also provide a corresponding proof when the condition of adherence is replaced by the requirement that the speed at a point on ∂D be an assigned function of the shear stress on the wall in the direction of flow at that point. We may state their result roughly as follows: *For a general incompressible simple fluid, the only tubes in which a steady nonvanishing rectilinear flow can adhere to the wall are circular, with or without a concentric circular core.* Inspired by the work of Fosdick and Serrin, McLeod[6] studied the problem again by relaxing some of their restrictions.

Ericksen's analysis made it natural to conjecture that if an incompressible fluid were to be forced steadily down a tube having a cross-section for which no steady rectilinear flow exists, some other steady flow would occur. A departure from a classical streamline pattern is generally described as a "secondary flow." The secondary flow in this case is given by a component of velocity normal to the generators of the pipe, as a result of which the fluid points move along spiraliform streamlines. Green and Rivlin, as soon as they had seen Ericksen's analysis, exhibited a secondary flow down a pipe of elliptical cross-section by use of a particular fluid.

We may see in advance that calculation of such a flow will be intricate. Indeed, if we assume that the shear-stress function τ and the normal-stress function σ_1 may be expanded in series in κ, e.g. (5.3-3), then (13) is always satisfied to the second order as $\kappa \to 0$. The effect of incompatibility, then, must be of at least third order in some parameter whose smallness keeps the shearings small. No general method of solving the problem is now known. In Chapter 6 we shall calculate a solution for a certain special class of fluids after assuming that it exists and can be expressed as a power series in a certain parameter that may be interpreted as the specific driving force. We shall see that a secondary flow does indeed result, in general, and is of the order a^4.

[6] J.B. McLeod, "Overdetermined systems and the rectilinear steady flows of simple fluids," *Proceedings of a Conference on Partial Differential Equations, Springer Lecture Notes in Mathematics* no. 145, (1974) 193–204.

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6

Some Flows of Particular Nonlinear Fluids

6.1 Rivlin-Ericksen Fluids

The Principle of Local Action asserts that the stress at the body point X is unaffected at the time t by the history of the motion at other body points except those in some arbitrarily small neighborhood of $\chi(X, t)$, but it allows influence to arbitrarily long-past time. Thus, in general, a material point may have an arbitrarily long memory. In viscometric flows (Chapter 5) and, more generally, in monotonous motions (4.2-3), any such memory is given scant opportunity to make itself known, and for this reason many special problems regarding such flows are amenable to an easy solution. There is a second way to find tractable problems: instead of specializing the motion, specialize the material. Because of the obvious difficulty introduced by long-range memory, it is natural to propose for study a class of materials in which the stress at X is affected by the history of the motion only within an arbitrarily short interval $[t - \delta, t]$ of preceeding time, where δ is some positive number. Materials of this kind have *infinitesimal memory*. The history of the motion before any given past time is irrelevant in determining the stress in such a material at the present time.

The most commonly studied materials with infinitesimal memory are simple materials in which the stress at X is determined by the first n derivatives of the local deformation $\mathbf{F}$ at the reference place X. Such a material is called a material of the *differential type*, and its complexity is called n. The defining constitutive relation for such a material is

$$\mathbf{T} = \mathbf{f}\left(\mathbf{F}, \dot{\mathbf{F}}, \ldots, \overset{(n-1)}{\mathbf{F}}, \overset{(n)}{\mathbf{F}}\right). \tag{6.1-1}$$

The $n + 1$ tensor arguments are mapped by $\mathbf{f}$ onto symmetric tensors, and the arguments X and t are not written.

The principle of material frame indifference leads to the following *reduced form* for the constitutive relation of a material of the differential type of complexity n (see Sections 2.2 and 2.3);

$$\mathbf{R}^T\mathbf{T}\mathbf{R} = \mathbf{g}(\mathbf{R}^T\mathbf{A}_1\mathbf{R}, \mathbf{R}^T\mathbf{A}_2\mathbf{R}, \ldots, \mathbf{R}^T\mathbf{A}_n\mathbf{R}, \mathbf{C}); \tag{6.1-2}$$

where $\mathbf{g}$ maps $n + 1$ symmetric tensor arguments onto a symmetric tensor and $\mathbf{R}$ is the rotation tensor from the reference placement adopted.

EXERCISE 6.1.1 (Noll)
Show that the reduced form (2) follows either by specializing (Sections 4.1–4.3) appropriately or by direct appeal to the principle of material frame indifference.

When the material of differential type is isotropic, it is called a *Rivlin-Ericksen solid* or *Rivlin-Ericksen fluid* of complexity n, according to whether its peer group is the orthogonal group $\mathcal{O}$ for some reference placement or the unimodular group $\mathcal{U}$ for all reference placements, in conformity with the definitions laid down in Chapter 2. The constitutive relations for these two instances are

$$\mathbf{T} = \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \mathbf{B}) \tag{6.1-3}$$

and

$$\mathbf{T} = \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \rho), \tag{6.1-4}$$

respectively. The former holds only if the Cauchy-Green tensor $\mathbf{B}$ is calculated with respect to an undistorted placement of the solid, while the latter holds in general. In both, the function $\mathbf{f}$ is *isotropic* in the sense that

$$\begin{aligned}\mathbf{f}(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T, \mathbf{Q}\mathbf{A}_2\mathbf{Q}^T, \ldots, \mathbf{Q}\mathbf{A}_n Q^T, \mathbf{Q}\mathbf{B}\mathbf{Q}^T \text{ or } \rho)\\ = \mathbf{Q}\mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \mathbf{B} \text{ or } \rho)\mathbf{Q}^T\end{aligned} \tag{6.1-5}$$

for all arguments of $\mathbf{F}$ and for all orthogonal tensors $\mathbf{Q}$. It is a routine matter to write out corresponding conclusions for incompressible materials.

EXERCISE 6.1.2
Show that the reduced forms (3), (4), and (5) follow either by specializing (4.1-2) and (4.1-3) or by a direct analysis of (2). Also, show that for a fluid in rigid motion, $\mathbf{f}(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \rho)$ is a spherical tensor.

The general solution **f** of the functional equation (5) is known,[1] but we shall need only its application to a fluid of complexity 2:

$$\begin{aligned}\mathbf{T} = & -p\mathbf{1} + \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_1^2 + \alpha_4\mathbf{A}_2^2 + \alpha_5(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1)\\ & + \alpha_6(\mathbf{A}_1^2\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1^2) + \alpha_7(\mathbf{A}_1\mathbf{A}_2^2 + \mathbf{A}_2^2\mathbf{A}_1) + \alpha_8(\mathbf{A}_1^2\mathbf{A}_2^2 + \mathbf{A}_2^2\mathbf{A}_1^2); \quad (6.1\text{-}6)\end{aligned}$$

the coefficients $p, \alpha_1, \alpha_2, \ldots, \alpha_8$ are functions of the density and the following scalar invariants of $\mathbf{A}_1$ and $\mathbf{A}_2$:

$$\begin{aligned}&\operatorname{tr}\mathbf{A}_1, \operatorname{tr}\mathbf{A}_1^2, \operatorname{tr}\mathbf{A}_1^3, \operatorname{tr}\mathbf{A}_2, \operatorname{tr}\mathbf{A}_2^2, \operatorname{tr}\mathbf{A}_2^3,\\ &\operatorname{tr}\mathbf{A}_1\mathbf{A}_2, \operatorname{tr}\mathbf{A}_1\mathbf{A}_2^2, \operatorname{tr}\mathbf{A}_1^2\mathbf{A}_2, \operatorname{tr}\mathbf{A}_1^2\mathbf{A}_2^2. \quad (6.1\text{-}7)\end{aligned}$$

The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_8$ are not uniquely determined, but in general there is no relation giving any one of them as a function of the others. For an incompressible fluid, conclusions of the same form hold, but p is then arbitrary, as are $\operatorname{tr}\mathbf{A}_1 = 0$ and $\operatorname{tr}\mathbf{A}_1^2 = \operatorname{tr}\mathbf{A}_2$, so the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_8$ are functions of only 8 rather than 10 scalar arguments.[2]

It follows from (4.3-5) that in a viscometric flow the response of a simple fluid cannot be distinguished from that of an appropriately selected Rivlin-Ericksen fluid of complexity 2. The following exercise enlarges upon this fact and shows that the viscometric functions fail to determine a unique Rivlin-Ericksen fluid of complexity 2. Thus the behavior of such a fluid in viscometric flows does not determine its constitutive equation uniquely.

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[1] There is an extensive literature on representation theorems, most of it on the assumption that **f** is a polynomial. Rivlin and Ericksen proved some particular theorems without such a restriction. Fully general algebraic representations for scalar-valued, vector-valued, and tensor-valued isotropic functions of any finite number of vectors, symmetric tensors, and skew tensors are given by C.-C Wang, "A new representation theorem for isotropic functions," *Archive for Rational Mechanics and Analysis* 36 1970: 166–223.

[2] The members of a set of tensors generally satisfy polynomial identities; also, as the space of symmetric tensors over a three-dimensional vector space is six-dimensional, any set of seven or more symmetric tensors is linearly dependent. These facts do not permit us, in general, to express any one of such a set of tensor *variables* as a function of the others. A simple example, namely,

$$(\operatorname{tr}\mathbf{A})\mathbf{B} = \mathbf{A},$$

suffices to show that a relation between two tensors **A** and **B** does not generally imply that one is a function of the other. First, **A** cannot here generally be a function of **B**, since if **A** is a solution corresponding to **B**, so is C**A** for any number C. On the other hand, **B** cannot here generally be a function of **A**, since any **B** is a solution corresponding to $\mathbf{A} = 0$. In all cases, nonetheless, either $\mathbf{B} = \mathbf{f}(\mathbf{A})$ or $\mathbf{A} = \mathbf{g}(\mathbf{B})$. Indeed, if $\operatorname{tr}\mathbf{A} \neq 0$, then $\mathbf{B} = (\operatorname{tr}\mathbf{A})^{-1}\mathbf{A}$; while if $\operatorname{tr}\mathbf{A} = 0$, it follows that $\mathbf{A} = 0$, whatever **B** is. Thus, if we consider the restriction of some function $\mathbf{f}(\mathbf{A}, \mathbf{B})$ to arguments such that $(\operatorname{tr}\mathbf{A})\mathbf{B} = \mathbf{A}$, we can always without loss of generality eliminate one or the other of the variables **A** and **B**, but not always the same one.

EXERCISE 6.1.3 (Rivlin, Markovitz)
In a viscometric flow the functions α_r of the ten arguments (7) are equal to certain functions of the shearing κ; each $\bar{\alpha}_r$ is an even function; show that the viscometric functions of the fluid are related as follows to the functions $\bar{\alpha}_r$:

$$\begin{aligned}\mu &= \bar{\alpha}_1 + 2\kappa^2\bar{\alpha}_5 + 4\kappa^4\bar{\alpha}_7,\\ \sigma_1 &= \kappa^2(2\bar{\alpha}_2 + \bar{\alpha}_3 + r\kappa^2\bar{\alpha}_4 + 4\kappa^2\bar{\alpha}_6 + 8\kappa^4\bar{\alpha}_8),\\ \sigma_2 &= \kappa^2\bar{\alpha}_3.\end{aligned} \tag{6.1-8}$$

To represent the general behavior of simple fluids in viscometric flows, it suffices to consider just the Rivlin-Ericksen fluids of complexity 2 for which $\alpha_4 = \alpha_5 = \cdots = \alpha_8 = 0$ and the coefficients α_1, α_2, and α_3 are functions of only $\operatorname{tr} A_1^2$.

When we come to study motions that are not viscometric flows, then, as we remarked at the beginning of this section, we shall generally be forced to assume the fluid devoid of long-range memory. Nevertheless, even the Rivlin-Ericksen fluids, as we can see from the example (6) with its eight material functions, will give rise to great mathematical difficulties. If we specialize the constitutive relation still further, we run the danger of eliminating whatever phenomenon we wish to study. Consider, for example, the particular fluid of complexity 2 mentioned at the end of Exercise 1.3. This fluid suffices for absolute generality in response to viscometric flows. If we go one step further and suppose that $2\alpha_2 + \alpha_3$ is proportional to α_1, the condition (5.7-13) is satisfied, and rectilinear flow is possible in pipes of all cross-sections. Such is the case, in particular, if we take the functions α_1, α_2, and α_3 as reducing to constants.

These considerations suggest the need for some systematic method of classifying Rivlin-Ericksen fluids into categories of decreasing similarity of response. There are many such systems. Here we shall consider one based on a formal expansion procedure. These results have a certain status in terms of a theorem of approximation, but for the time being we shall look at it in a purely formal way.

We suppose now that one particular motion $\chi(X, t)$ of a body be given, and from it, by retarding the time scale by a *positive retardation factor* r, we construct a one-parameter family of *retarded motions*:[3]

$$\operatorname{ret}\chi(X, t) := \chi(X, rt). \tag{6.1-9}$$

That is, if the body point X was at the place $\mathbf{x}$ at the time t in the motion χ, in the retarded motion $\operatorname{ret}\chi$ it did not reach that same place until the time t/r. In the interpretation, we shall think of (9) in the limit as $r \to 0$, so that in $\operatorname{ret}\chi$ the

[3]Coleman and Noll's theory of retardation applied to simple materials in general provides an expansion in terms of deformation histories described more and more slowly. WANG's theories of relaxations and retardations provide expansions in terms of families of materials. For a brief description of these see section 40, C Truesdell and W. Noll, *The Non-Linear Field Theories of Mechanics*, (Berlin: Springer-Verlag, 1992).

body point X will move along just the same path as in the motion χ but at a rate uniformly slower in the ratio r.

If we attach to functions other than χ the prefix ret to denote quantities calculated from ret χ, for the nth velocity $\overset{(n)}{\mathbf{x}}$ we shall have

$$\operatorname{ret} \overset{(n)}{\chi} := \partial_t^n \operatorname{ret} \chi = r^n \overset{(n)}{\chi}, \tag{6.1-10}$$

and hence by (2.2-9),

$$\operatorname{ret} \mathbf{G}_n = r^n \mathbf{G}_n. \tag{6.1-11}$$

From (11) and (2.2-17), we obtain for the Rivlin-Ericksen tensors ret$\mathbf{A}_n$ of the retarded motion:

$$\operatorname{ret} \mathbf{A}_n = r^n \mathbf{A}_n. \tag{6.1-12}$$

Returning to the constitutive equation (4) of a Rivlin-Ericksen fluid of complexity n, we specialize it by the following further assumptions.

1. The function $\mathbf{f}$ is a polynomial in the arguments $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$.

2. In the retarded motion retχ the function $\mathbf{f}$ is a polynomial of degree n in the retardation factor r. The special fluid of complexity n so defined is called the *fluid of grade n*. If we think of a motion χ as given and fixed, and if we construct from it the retarded motions retχ, then as $r \to 0$, the constitutive equation for the fluid of grade n ultimately approximates that of the general fluid of complexity n to within an error $\mathrm{O}(r^{n+1})$, provided we adopt in the first place the assumption that $\mathbf{f}$ is a polynomial.

For example, we see from (6) that while the constitutive equation for the fluid of complexity 1 is

$$\mathbf{T} = -p\mathbf{1} + \alpha_1 \mathbf{A}_1 + \alpha_3 \mathbf{A}_1^2, \tag{6.1-13}$$

in which p, α_1, and α_3 are arbitrary functions of $\operatorname{tr} \mathbf{A}_1$, $\operatorname{tr} \mathbf{A}_1^2$, $\operatorname{tr} \mathbf{A}_1^3$, and r, the constitutive relation for the fluid of *grade* 1, since it must be of degree 1 in r in the retarded motion retχ, by (12) is

$$\mathbf{T} = \left(-p + \frac{1}{2}\lambda \operatorname{tr} \mathbf{A}_1\right)\mathbf{1} + \mu \mathbf{A}_1, \tag{6.1-14}$$

in which p, λ, and μ are functions of ρ only. If the fluid is incompressible, (14) is replaced by

$$\mathbf{T} = -p\mathbf{1} + \mu \mathbf{A}_1; \tag{6.1-15}$$

p is indeterminate in the sense that it is not determined by the history of the transplacement gradient and does not determine it, and μ is a constant, the former for an unconstrained fluid and the latter for an incompressible one. These two formulae are the constitutive relations of the Navier-Stokes fluids. Thus *the fluid of grade 1 is the Navier-Stokes fluid, while the elastic fluid is the fluid of grade 0.*

In the same way we can write down from (6) and (7) the constitutive relation of the unconstrained fluid of grade 2:

$$\mathbf{T} = \left(-p + \tfrac{1}{2}\boldsymbol{\lambda} \operatorname{tr} \mathbf{A}_1 + \alpha_{10} \operatorname{tr} \mathbf{A}_2 + \alpha_{20} \operatorname{tr} \mathbf{A}_1^2 + \alpha_{30}(\operatorname{tr} \mathbf{A}_1)^2\right) \mathbf{1} \\ + (\mu + \alpha_{11} \operatorname{tr} \mathbf{A}_1)\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \qquad (6.1\text{-}16)$$

where $p, \boldsymbol{\lambda}, \mu, \alpha_{10}, \alpha_{20}, \alpha_{30}, \alpha_{11}, \alpha_1$, and α_2 are functions of ρ. In an isochoric motion, the terms whose coefficients are $\boldsymbol{\lambda}$, α_{30}, and α_{11} vanish; for an incompressible fluid the stress is indeterminate to within an arbitrary hydrostatic pressure p, for such a fluid the terms whose coefficients are α_{10} and α_{20} can be absorbed in p, so the constitutive relation of the incompressible fluid of grade 2 is simply

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \qquad (6.1\text{-}17)$$

where μ, α_1, and α_2 are constants.

The procedure that delivered (15) and (17) by use of the retardation (9) began with the representation (6) of a fluid of complexity 2. For use in Section 2 we need the constitutive equation for the determinate stress of an incompressible fluid of grade 4. For it Rivlin obtained the following formula,

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4, \qquad (6.1\text{-}18)$$

in which, after removal of a redundancy, in $\mathbf{S}_4$,

$$\begin{aligned} \mathbf{S}_1 &= \mu_o\mathbf{A}_1, \\ \mathbf{S}_2 &= \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \\ \mathbf{S}_3 &= \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(\operatorname{tr} \mathbf{A}_2)\mathbf{A}_1, \\ \mathbf{S}_4 &= \gamma_1\mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) \\ &\quad + \gamma_5(\operatorname{tr} \mathbf{A}_2)\mathbf{A}_2 + \gamma_6(\operatorname{tr} \mathbf{A}_2)\mathbf{A}_1^2 \\ &\quad + [\gamma_7 \operatorname{tr} \mathbf{A}_3 + \gamma_8 \operatorname{tr}(\mathbf{A}_2\mathbf{A}_1)]\mathbf{A}_1 \end{aligned} \qquad (6.1\text{-}19)$$

where $\mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \ldots, \gamma_8$ are constants. As we shall see below in *Exercise VI.1.4.*, μ_0 is the shear viscosity constant of the fluid, but it cannot generally be identified with the shear viscosity function of a fluid of grade n if $n > 2$.

In the formulae just given,

$$\mathbf{S}_k = \mathbf{O}(r^k) \text{ as } r \to 0 \qquad (6.1\text{-}20)$$

in the retarded motions retχ. Thus, for example, $\mathbf{S}_3$ is exactly what must be added to $\mathbf{S}_1 + \mathbf{S}_2$ in order to obtain from the constitutive relation of the fluid of grade 2 that of the fluid of grade 3.

EXERCISE 6.1.4

From (4.2-15) it follows that in a monotonous flow we may use the modulus $|\kappa|$ of the shearing as the retardation factor r. In a viscometric flow

$$\begin{aligned}
\mathbf{S}_1 &= \mu_0\kappa(\mathbf{N}+\mathbf{N}^T),\\
\mathbf{S}_2 &= \kappa^2[2\alpha_1\mathbf{N}^T\mathbf{N}+\alpha_2(\mathbf{N}+\mathbf{N}^T)^2],\\
\mathbf{S}_3 &= 2\kappa^3(\beta_2+\beta_3)(\mathbf{N}+\mathbf{N}^T),\\
\mathbf{S}_4 &= \kappa^4[4(\gamma_3+\gamma_4+\gamma_5)\mathbf{N}^T\mathbf{N}+2\gamma_6(\mathbf{N}+\mathbf{N}^T)^2],
\end{aligned} \tag{6.1-21}$$

while the viscometric functions are given by

$$\begin{aligned}
\mu &= \mu_0+2(\beta_2+\beta_3)\kappa^2,\\
\sigma_1 &= (2\alpha_1+\alpha_2)\kappa^2+[4(\gamma_3+\gamma_4+\gamma_5)+2\gamma_6]\kappa^4,\\
\sigma_2 &= \alpha_2\kappa^2+2\gamma_6\kappa^4.
\end{aligned} \tag{6.1-22}$$

In general, the viscometric functions of a fluid of grade n are polynomials of degree at most n in κ. Since according to the general theory of fluids the viscometric functions need not be polynomials at all, for no n does the theory of fluid of grade n cover all possibilities in viscometric flow. This fact should serve to distinguish clearly the difference between "grade" and "complexity," since, as we have seen in Exercise 5.1.1. in Chapter 5, the theory of the fluid of complexity 2 covers the fully general theory of viscometric flows. Both "grade" and "complexity" suggest the outcome of a process of approximation; the lower the complexity of the fluid, the lower the order of derivatives of the velocity field needed to determine the stress in it, while the lower the grade of the fluid, the slower the flow described adequately by its constitutive relation. On the other hand, we must remember that the approximative processes are merely suggestive, not proved, and that they are not necessary in order for us to consider the fluid of grade n or of complexity n, since such a fluid satisfies the general requirements of continuum mechanics and may be an object of study in itself. In particular, the Navier-Stokes fluid and the elastic fluid, which are the fluids of grades 1 and 0, respectively, do not have to be considered as approximations to anything more general but deserve analysis as independent objects, specimens of what a fluid might be. Classical hydrodynamics, which usually limits attention to these two fluids, is thus an exact theory, though a special one.

By specialization of (22) we see that the viscometric functions of the fluid of grade 2 are given by

$$\begin{aligned}
\mu &= \mu_0,\\
\sigma_1 &= (2\alpha_1+\alpha_2)\kappa^2,\\
\sigma_2 &= \alpha_2\kappa^2.
\end{aligned} \tag{6.1-23}$$

Comparison with (17) shows that if a set of three viscometric functions is compatible with the theory of fluids of grade 2, it determines such a fluid uniquely. *The constitutive relations of an incompressible fluid of grade 2 is thus determined by the* *behavior of that fluid in viscometric flows.* This conclusion holds *a fortiori* for the Navier-Stokes fluid. Viscometric problems were first considered mainly because outcomes of viscometric experiments were thought to suffice for determining all the physical characteristics of an incompressible fluid, so that, in principle at least, viscometric data should suffice as the basis for the subsequent prediction of all behavior of the fluid by mathematical process alone. A glance at (22) shows that such is not generally the case. For example, the viscometric functions of the fluid of grade 3 reveal nothing at all about β_1, and they determine nothing more about β_2 and β_3 than their sum. Indeed, in $(19)_3$ the coefficient β_1 multiplies the tensor $\mathbf{A}_3$, which vanishes in a viscometric flow, and so in principle a flow in which $\mathbf{A}_3 \neq 0$ has to be considered if the value of β_1 is to matter at all. These observations illustrate the inherent limitations of the fluid of grade 2 and of its special case, the Navier-Stokes fluid.

EXERCISE 6.1.5 (Berker)
Use of (2.2-17), (2.2-19), and (2.2-21) shows that at all points on a stationary wall to which a body adheres,

$$\begin{aligned} \mathbf{A}_1\mathbf{n} &= 2E\mathbf{n} + 2\mathbf{W}\mathbf{n}, \\ \mathbf{A}_1^2\mathbf{n} &= (4E^2 + \mathrm{w}^2)\mathbf{n} + 4E\mathbf{W}\mathbf{n}, \\ \mathbf{A}_2\mathbf{n} &= (2E' + 4E^2 + 2\mathrm{w}^2)\mathbf{n} + 2E\mathbf{W}\mathbf{n} + 2\mathbf{W}'\mathbf{n}, \end{aligned} \tag{6.1-24}$$

where E is the expansion, $\mathbf{W}$ the spin, and w the magnitude of the vorticity defined by $\omega := |\operatorname{curl} \dot{\mathbf{x}}|$. The axis of spin needs not be steady. The traction exerted by the wall upon an adherent body of incompressible fluid of grade 2 is given by

$$\begin{aligned} \mathbf{t} &= [-p + (2\alpha_1 + \alpha_2)\mathrm{w}^2]\mathbf{n} + 2\mu\mathbf{W}\mathbf{n} + 2\alpha_1\mathbf{W}'\mathbf{n} \\ &= [-p + (2\alpha_1 + \alpha_2)\mathrm{w}^2]\mathbf{n} - \mu\mathrm{w}\mathbf{f} - \alpha_1(\mathrm{w}\mathbf{f})', \end{aligned} \tag{6.1-25}$$

where $\mathbf{f}$ is a unit vector in the direction obtained by rotating the vorticity vector through a right angle counterclockwise about $\mathbf{n}$.

Berkers's formula (25) further illustrates the special nature of the incompressible fluid of grade 2. The shear traction exerted at a place on a stationary wall by such a fluid in steady flow is $2\mu\mathbf{W}\mathbf{n}$, just the same as would be exerted by a Navier-Stokes fluid of the same shear viscosity μ in undergoing a flow having the same spin $\mathbf{W}$ at that place. Of course it does not follow that in the solution of some boundary-value problem the shear traction really is unaffected by the values of α_1 and α_2, for those constants enter the equations of motion, solutions of which, for given boundary conditions, may deliver a spin field different from that provided by the Navier-Stokes equations.

The general principles set forth in this book leave many constants and functions arbitrary. For example, in the constitutive relation (14), that defines the Navier-

Stokes theory of fluids, the functions λ and μ are not restricted. Nevertheless, it is natural to expect that viscosity shall be a dissipative phenomenon, in which work may be consumed in the interior of a body but cannot be created there. Specifically, the power of the portion of the stress that depends upon $\mathbf{D}$ should not be negative for any $\mathbf{D}$. This adscititious requirement, which was imposed by Duhem and Stokes, is easily shown to be equivalent to the inequalities

$$\mu > 0, \quad 3\lambda + 2\mu > 0. \tag{6.1-26}$$

Nowadays it is customary to derive this restriction and others of the same kind by imposing the second law of thermodynamics as an identical requirement upon constitutive mappings. In this textbook thermodynamic principles are not taken up. We shall continue to leave constitutive constants and functions unrestricted except by considerations arising in pure mechanics, and we shall not assume that the stress power cannot ever be negative.

6.2 Boundary Conditions and Perturbation Procedures

In the applications of the Navier-Stokes theory, the fluid body is usually assumed to adhere to rigid boundaries, and there are existence theorems that provide unique solutions to various problems presuming that condition. Following that precedent, most studies of the motions of nonlinear fluids also adopt the condition of adherence and attempt to make it seem sufficient to deliver unique solutions. Indeed, while that is sometimes possible, generally it is not.

Consider, for example, the incompressible fluid of grade 2, for which

$$\mathbf{S}_2 = \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2. \tag{6.2-1}$$

From (2.2-18) we know that

$$\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{G} + \mathbf{G}^T\mathbf{A}_1 = \mathbf{A}_1' + (\operatorname{grad}\mathbf{A}_1)\dot{\mathbf{x}} + \mathbf{A}_1\mathbf{G} + \mathbf{G}^T\mathbf{A}_1.$$

Thus the term $(\operatorname{grad}\operatorname{grad}\dot{\mathbf{x}})\dot{\mathbf{x}}$ appears in $\mathbf{A}_2$ and hence in $\mathbf{S}_2$, and therefore the term $\operatorname{div}[(\operatorname{grad}\operatorname{grad}\dot{\mathbf{x}})\dot{\mathbf{x}}]$ appears in $\operatorname{div}\mathbf{S}_2$. Therefore the term in the partial differential equation of motion that arises from $(\operatorname{grad}\mathbf{A}_1)\dot{\mathbf{x}}$ is linear in the third spatial derivatives of $\dot{\mathbf{x}}$, while the Navier-Stokes equation involves only the second spatial derivatives of $\dot{\mathbf{x}}$, in which it is linear. This fact suggests that to obtain unique solutions in the theory of fluids of grade 2, in general some boundary condition in addition to adherence should generally be supplied. What that condition is, we do not know. It should reflect some physical idea, but none appropriate has been put forward. However for special flows, e.g., flows in bounded domains with homogeneous boundary conditions, such results have been obtained.

The same difficulty arises, indeed is magnified, for fluids of greater grade. As is suggested by (4.1-19), $\mathbf{S}_n$ includes a term proportional to $\mathbf{A}_n$; $\mathbf{A}_n$ depends linearly on $\dot{\mathbf{A}}_{n-1}$; a summand in $(\mathbf{A}_{n-1})$ is $(\text{grad}\,\mathbf{A}_{n-1})\dot{\mathbf{x}}$. The partial differential equation of motion for a fluid of grade n, therefore, will generally be of order $n+1$ in the spatial derivatives of $\dot{\mathbf{x}}$.

In the simplest problems, some of which we have studied and others of which we shall study

$$(\text{grad}\,\mathbf{A}_{n-1})\dot{\mathbf{x}} = \mathbf{0}, \quad n = 2, 3, \ldots, \tag{6.2-2}$$

and the difficulty does not arise.

Also some progress has been made with problems concerning infinite domains by assuming boundedness of solutions and sufficiently rapid decay of derivatives at infinity. Although that is a common analytic device, it is something that should be proved rather than assumed.

A device of a different kind has been used in the literature of rheology. Namely, the constitutive equation of an incompressible fluid has been transmogrified into

$$\mathbf{S} = \mu\mathbf{A}_1 + \varepsilon\mathbf{g}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n), \tag{6.2-3}$$

in which $\mathbf{g}$ is an isotropic tensor polynomial function and ε is a "small" parameter. Conclusions from perturbations obtained by expansions in powers of ε are named appropriate to a "nearly Newtonian" or "slightly visco-elastic fluid." Students will see at once that if $n > 2$, this procedure as $\varepsilon \to 0$ wipes out the spatial derivatives beyond those appearing in the Navier-Stokes equation. To make matters worse, inertia is neglected at every stage. The same is true of later procedures in which by assumption $\dot{\mathbf{x}} = \varepsilon\mathbf{u}$ for a fixed flow $\mathbf{u}$ and all quantities arising in the attempt to satisfy the equations of motion become polynomials in ε, which is assumed small. The mathematical problem is one of singular perturbation, but the procedure applied is one of ordinary perturbation.

Sometimes correct conclusions may be obtained by such unconvincing methods. Indeed, acceptable statements about the flow of incompressible fluids down straight pipes of noncircular cross-section were first obtained in this way, but here, following Noll, we shall obtain them by *power series* of a presumed solution of the actual problem.

6.3 Secondary Steady Flow of a Simple Fluid down a Straight Pipe: Preliminaries

As we showed in Section 5.5, most fluids cannot flow steadily straight down a straight pipe unless the cross-section is of a special kind; only certain special fluids may flow steadily straight down a straight pipe of general cross-section. Both the

Navier-Stokes fluid and the fluid of grade 2 are exceptional in this sense, since their viscometric functions satisfy the special relation (5.7-13). If a general fluid is forced by a steady pressure gradient down a pipe bounded by a cylindrical surface that is neither a circular cylinder nor a pair of coaxial circular cylinders, we expect that a steady flow shall ensue, but it cannot be rectilinear. The component of the velocity field normal to the generators of the cylindrical boundary is called a *secondary flow*. The simplest velocity fields of this kind seem to be those specified as follows in terms of the position vector $\mathbf{p}$ in the plane cross-section:

$$\dot{\mathbf{x}} = \mathrm{v}(\mathbf{p})\mathbf{k} + \mathbf{u}(\mathbf{p}), \;\; \operatorname{div}\mathbf{u} = 0, \dot{\mathbf{x}} = 0 \text{ if } \mathbf{p} \in \partial\mathcal{A}. \qquad (6.3\text{-}1)$$

Thus a solution is sought in which the primary flow is $\mathrm{v}(\mathbf{p})$ in the direction of the unit vector $\mathbf{k}$ normal to and the secondary flow $\mathbf{u}(\mathbf{p})$ is the same at each cross-section. We consider (1) as expressing a semiinverse hypothesis, and we shall show that suitable selection of $\mathrm{v}(\mathbf{p})$ leads to a formal power series in a specific, appropriate parameter provided by the problem itself.

We assume that the flow domain $\mathcal{A}$ is simply connected.[4] Since $\operatorname{div}\mathbf{u} = 0$, there is a stream function q such that

$$\mathbf{u} = (\nabla q)^{\perp}, \;\; \mathbf{p} \in \partial\mathcal{A}. \qquad (6.3\text{-}2)$$

Here and henceforth we write ∇ for the spatial gradient in $\mathcal{A}$ and use the notation $\mathbf{a}^{\perp}$ for the vector obtained by rotating a vector $\mathbf{a}$ that is normal to $\mathbf{k}$ counterclockwise through a right angle about $\mathbf{k}$ i.e., $\mathbf{a}^{\perp} = \mathbf{k} \times \mathbf{a}$. Thus $(\mathbf{a}^{\perp})^{\perp} = -\mathbf{a}$. On $\partial\mathcal{A}$ the boundary condition expressing adherence of the fluid is

$$q = \text{const.}, \;\; \partial_{\mathrm{n}} q = 0. \qquad (6.3\text{-}3)$$

where ∂_{n} is the normal derivative; without loss of generality we may and shall replace the first formula by $q = 0$.

From (1) we see that

$$\mathbf{G} = \operatorname{grad}\,\dot{\mathbf{x}} = \mathbf{k} \otimes \nabla\mathrm{v} + \nabla\mathbf{u}; \qquad (6.3\text{-}4)$$

hence

$$\ddot{\mathbf{x}} = \mathbf{G}\dot{\mathbf{x}} = (\mathbf{u} \cdot \nabla\mathrm{v})\mathbf{k} + (\nabla\mathbf{u})\mathbf{u}. \qquad (6.3\text{-}5)$$

The nth velocity $\overset{(n)}{\mathbf{x}}$, its gradient $\mathbf{G}_n$, and the nth Rivlin-Ericksen tensor $\mathbf{A}_n$ are steady and independent of the cross-section. Hence

$$(\nabla\mathbf{A}_n)\mathbf{k} = \mathbf{0}, \qquad (6.3\text{-}6)$$

and the recurrence formula (2.2-18) for the Rivlin-Ericksen tensors reduces to

$$\mathbf{A}_{n+1} = (\nabla\mathbf{A}_n)\mathbf{u} + \mathbf{A}_n\mathbf{G} + (\mathbf{A}_n\mathbf{G})^T. \qquad (6.3\text{-}7)$$

Because the secondary motion is assumed to be the same at all cross-sections, so also is the determinate stress of an incompressible fluid of any grade, and, by

[4]This assumption is unduly strong and can be relaxed.

(5), so also is the acceleration $\ddot{\mathbf{x}}$. If we write z for a coordinate in the direction of $\mathbf{k}$, then by differentiating Cauchy's first law in the form (5.4-1) with respect to z we obtain

$$\partial_z \text{ grad } \varphi = 0. \tag{6.3-8}$$

Integration of (8) shows that

$$\rho\varphi = -az + \zeta(p); \tag{6.3-9}$$

here a is a constant, and the function ζ remains to be determined. By recalling the definition of φ in (3.7-5) and assuming that ϖ is independent of z, we see from (9) that $\partial_z p = -a$, so the *specific driving force a* is the only agent that pushes the fluid through the pipe.

To discuss the remaining implications of Cauchy's first law, we employ the following decomposition of the determinate stress at $\mathbf{x}$:

$$\mathbf{S} = N(\mathbf{k} \otimes \mathbf{k}) + \mathbf{t} \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{t} + \mathbf{\Pi}, \tag{6.3-10}$$

in which N is a scalar, $\mathbf{t}$ is a vector normal to $\mathbf{k}$, and $\mathbf{\Pi}$ is a tensor over the two-dimensional space of vectors normal to $\mathbf{k}$, and all three summands are presumed to vary smoothly with $\mathbf{x}$. By use of (10) and (9) we see that

$$T_{zz} = -p + N(\mathbf{p}) = N(\mathbf{p}) + az - \zeta(\mathbf{p}) = \rho\varpi(\mathbf{p}), \tag{6.3-11}$$

while by use of (5) Cauchy's first law (as expressed by (5.4-1)) becomes equivalent to

$$\begin{aligned} \operatorname{div} \mathbf{t} + a &= \rho\dot{\mathbf{u}} \cdot \nabla \mathrm{v} \\ \operatorname{div} \mathbf{\Pi} - \nabla\zeta &= \rho(\nabla\mathbf{u})\mathbf{u}. \end{aligned} \tag{6.3-12}$$

The essence of Noll's method is to use the driving force a, which arises in $(12)_1$ as a natural parameter for the problem, as the variable of a formal power-series expansion of a presumed solution of (12).

6.4 Calculation of the Series Determining Secondary Flow down a Straight Pipe

We choose as the fluid to undergo the flow some Rivlin-Ericksen fluid of grade n, n being fairly large and we assume that there are solutions for it expressible as follows for values of the driving force a close to zero:

$$\mathrm{v} = \sum_{r=1}^{4} a^r \mathrm{v}_r + O(a^5),$$

$$\mathbf{u} = \sum_{r=1}^{4} a^r \mathbf{u}_r + \mathbf{O}(a^5), \mathbf{u}_r = (\nabla q_r)^{\perp},$$

$$\zeta = \sum_{r=1}^{4} a^r \zeta_r + O(a^5). \tag{6.4-1}$$

The boundary conditions on discharge follow from (6.3-3):

$$\mathrm{v}_r = 0, \ q_r = 0, \partial_n q_r = 0, \ r = 1, 2, 3, 4. \tag{6.4-2}$$

Further, we assume that the derivatives with respect to $\mathbf{p}$ of the remainders written as $\mathbf{O}(a^5)$ or $\mathbf{O}(a^5)$ are also of those same orders.

It is plausible that the actual flow will be slow when a is small. However, it has not been proved here that the flow from a small specific driving force can be obtained from the flow for a larger specific driving force by a mere retardation. If there is such a solution, and if the response of that fluid is smooth, it differs from that of a fluid of grade n by a function that is $\mathbf{O}(a^n)$.

Proceeding with the expansion, from (6.3-4) we see that

$$\mathbf{G} = \sum_{r=1}^{4} a^r (\mathbf{k} \otimes \nabla \mathrm{v}_r + \nabla \mathbf{u}_r) + \mathbf{O}(a^5), \tag{6.4-3}$$

but calculation of $\mathbf{G}_n$ for values of n greater than 1 becomes elaborate, since repeated multiplication of series is necessary. From $(1)_2$, (3), and (6.3-7), we see that even the expression for $\mathbf{A}_2$ will be very long. For the present problem, fortunately, we shall not need all of it, and we shall be able to show by successive calculation that many terms vanish.

We notice first that from the result of putting (1) in general into (6.3-7) it is obvious that

$$\mathbf{A}_n = \mathbf{O}(a^n) \quad \text{as } a \to 0. \tag{6.4-4}$$

Thus as $a \to 0$ the Rivlin-Ericksen tensors vanish to at least the same order as they do in the family of retarded flows with retardation factor a, according to (6.1-12). Consequently the stress as given by the constitutive relation of the fluid of grade n differs from that of any Rivlin-Ericksen fluid by a stress that in the class of flows we study here is $\mathbf{O}(a^{n+1})$ or perhaps even smaller.

We now determine the coefficients in the expansions (1) by substitution of those expansions into the constitutive relation for a fluid of grade n, substituting the result into the equations of linear momentum (6.3-12) and then equating to zero the coefficients of the successive powers of a. At the rth step we obtain a system of partial differential equations and boundary conditions for determining the functions v_r and $\mathbf{u}_r$.

Step 6.1 (Navier-Stokes Solution). By (3) we see that

$$\mathbf{A}_1 = \mathbf{G} + \mathbf{G}^T = a[\mathbf{k} \otimes \nabla \mathrm{v}_1 + \nabla \mathrm{v}_1 \otimes \mathbf{k} + \nabla \mathbf{u}_1 + (\nabla \mathbf{u}_1)^T] + \mathbf{O}(a^2); \tag{6.4-5}$$

by $(6.1\text{-}19)_1$,

$$\mathbf{T} + p\mathbf{1} = \mu_0 \mathbf{A}_1 + \mathbf{O}(a^2). \tag{6.4-6}$$

Comparison of (5) and (6) with (6.3-10) shows that

$$\begin{aligned} \mathbf{t} &= a\mu_0 \nabla \mathrm{v}_1 + \mathbf{O}(a^2), \\ \mathbf{\Pi} &= a\mu_0 [\nabla \mathbf{u}_1 + (\nabla \mathbf{u}_1)^T] + \mathbf{O}(a^2). \end{aligned} \tag{6.4-7}$$

We assume, of course, that $\mu_0 \neq 0$. Since $\mathbf{u} \cdot \nabla \mathrm{v}_1 = O(a^2)$ and $(\nabla \mathbf{u})\mathbf{u} = \mathbf{O}(a^2)$, substitution of these statements into (6.3-12) and equating to 0 the coefficients of a in both members yields

$$\begin{aligned} \mu_0 \Delta \mathrm{v}_1 &= -1, \\ \mu_0 \operatorname{div}[\nabla \mathbf{u}_1 + (\nabla \mathbf{u}_1)^T] &= \nabla \zeta_1, \end{aligned} \tag{6.4-8}$$

where Δ is the operator div ∇. The first of these conditions is a Poisson equation; with the boundary condition $\mathrm{v}_1(\mathbf{p}) = 0$ when $\mathbf{p} \in \partial \mathcal{A}$, that equation determines a unique solution v_1 in $\mathcal{A}$. It is the exact and entire solution of the problem for the fluid of grade 1, namely, the Navier-Stokes solution, but we must still consider the consequences of $(8)_2$.

To solve $(8)_2$, we note first that if $q = q(\mathbf{p})$ and $\mathbf{u} = (\nabla q)^\perp$, then

$$\operatorname{div}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] = (\nabla \Delta q)^\perp. \tag{6.4-9}$$

By $(1)_3$, $\mathbf{u}_1 = (\nabla q_1)^\perp$, so $(8)_2$ becomes

$$\mu_0 (\nabla \Delta q_1)^\perp = \nabla \zeta_1. \tag{6.4-10}$$

If we apply the operation $\perp$ to this equation and take the divergence of the result, we obtain

$$\mu_0 \Delta \Delta q_1 = 0. \tag{6.4-11}$$

Being the biharmonic equation, (11) has a unique solution q_1 such that both q_1 and $\partial_n q_1$ vanish upon $\partial \mathcal{A}$. That solution is $q_1 \equiv 0$. Hence $\mathbf{u}_1 \equiv \mathbf{0}$, and by (10) we see that $\zeta_1 = \text{const}$.

At the first stage, then, to within error $\mathbf{O}(a^2)$ we obtain the Navier-Stokes solution for the problem:

$$\begin{aligned} \dot{\mathbf{x}} &= a v_1 \mathbf{k} + \mathbf{O}(a^2), \\ \rho \varphi &= -az + \text{const.} + O(a^2). \end{aligned} \tag{6.4-12}$$

Of course, had we chosen to regard the Navier-Stokes constitutive equation as exact, we should have found the error terms identically zero. In the more general theory of the fluid of grade n, which we consider here, it is evaluation of the error terms that will explain the secondary flow.

Step 6.2 (Solution for the Fluid of Grade 2).
EXERCISE 6.4.1
If

$$\kappa_1 := |\nabla v_1|, \tag{6.4-13}$$

from $(12)_1$, (6.3-4), and (6.3-7), show that

$$\begin{aligned} \mathbf{A}_1^2 &= a^2[(\nabla v_1 \otimes \nabla v_1) + \kappa_1^2(\mathbf{k} \otimes \mathbf{k})] + \mathbf{O}(a^3), \\ \mathbf{A}_2 &= 2a^2 \nabla v_1 \otimes \nabla v_1 + \mathbf{O}(a^3), \end{aligned} \tag{6.4-14}$$

and hence

$$\mu_0 \Delta v_2 = 0, \ \ \mu_0 \Delta\Delta q_2 = 0, \tag{6.4-15}$$

so

$$v_2 = 0, q_2 = 0, \mathbf{u}_2 = \mathbf{0} \tag{6.4-16}$$

and

$$\dot{\mathbf{x}} = a v_1 \mathbf{k} + \mathbf{O}(a^3). \tag{6.4-17}$$

The conclusion of this exercise shows that the Navier-Stokes velocity field emerges to within an error $\mathbf{O}(a^3)$. For the fluid of grade 2, as we remarked in Section 3, a rectilinear solution is possible because (5.7-13) is satisfied; indeed, the velocity field of the Navier-Stokes solution furnishes an exact solution also for the fluid of grade 2, the error terms in (14) and (17) being strictly $\mathbf{0}$ in this case. Normal tractions[5] not provided by the Navier-Stokes theory are required to produce this same flow in the fluid of grade 2.

Step 6.3 (Approximate Solution for the Fluid of Grade 3). In view of (16) we see from (3) that

$$\begin{aligned} \mathbf{G} &= a\mathbf{k} \otimes \nabla v_1 + \mathbf{O}(a^3), \\ \mathbf{A}_1 &= a[\mathbf{k} \otimes \nabla v_1 + \nabla v_1 \otimes \mathbf{k}] \\ &\quad + a^3[\mathbf{k} \otimes \nabla v_3 + \nabla \mathbf{u}_3 + \nabla v_3 \otimes \mathbf{k} + (\nabla \mathbf{u}_3)^T] + \mathbf{O}(a^4). \end{aligned} \tag{6.4-18}$$

If we substitute these formulae, $(14)_2$, and $(16)_2$ into (6.3-7) and recall that v_1 does not depend on z, we find that

$$\mathbf{A}_3 = \mathbf{O}(a^5). \tag{6.4-19}$$

Likewise, we can strengthen (14) to read

$$\begin{aligned} \mathbf{A}_1^2 &= a^2[(\nabla v_1 \otimes \nabla v_1) + \kappa_1^2(\mathbf{k} \otimes \mathbf{k})] + \mathbf{O}(a^4), \\ \mathbf{A}_2 &= 2a^2 \nabla v_1 \otimes \nabla v_1 + \mathbf{O}(a^4). \end{aligned} \tag{6.4-20}$$

[5] See A.C. Pipkin and R.S. Rivlin, "Normal stresses in flow through tubes of noncircular cross-section," *Zeitschrift für Angewandte Mathematik und Physik* 14 (1963): 738–42.

EXERCISE 6.4.2

Show that substitution of these formulae into (6.1-18) and comparing the outcome with (6.3-10) yields

$$\mathbf{t} = a\mu_0\nabla \mathrm{v}_1 + a^3[\mu_0\nabla \mathrm{v}_3 + 2(\beta_2+\beta_3)\kappa_1^2\nabla \mathrm{v}_1] + \mathbf{O}(a^4),$$
$$\mathbf{\Pi} = a^2(2\alpha_1+\alpha_2)(\nabla \mathrm{v}_1 \otimes \nabla \mathrm{v}_1) + a^3\mu_0[\nabla \mathbf{u}_3 + (\nabla \mathbf{u}_3)^T] + \mathbf{O}(a^4). \quad (6.4\text{-}21)$$

By (16) we see that $\mathbf{u} = \mathbf{O}(a^3)$, $\nabla \mathrm{v} = \mathbf{0}(a)$, so

$$\mathbf{u}\cdot\nabla \mathrm{v} = O(a^4), (\nabla\mathbf{u})\mathbf{u} = \mathbf{O}(a^6). \quad (6.4\text{-}22)$$

Thus, in the dynamical equations (6.3-12), the contributions of the right-hand sides are of the order of the error when we consider only terms of order up to 3 in r.

EXERCISE 6.4.3

Show that substitution of (21) and (22) into (6.3-12) yields

$$\mu_0\Delta \mathrm{v}_3 = -2(\beta_2+\beta_3)\,\mathrm{div}(|\nabla \mathrm{v}_1|^2\nabla \mathrm{v}_1),$$
$$\mu_0\,\mathrm{div}[\nabla\mathbf{u}_3 + (\nabla\mathbf{u}_3)^T] = \nabla\zeta_3. \quad (6.4\text{-}23)$$

Steps parallel to those leading from $(8)_2$ to (11) show from $(23)_2$ that

$$\mu_0\Delta\Delta q_3 = 0. \quad (6.4\text{-}24)$$

Since $(23)_1$ is a Poisson equation, it has a unique solution v_3 determined by the boundary condition $\mathrm{v}_3(\mathbf{p}) = 0$ when $\mathbf{p} \in \partial\mathcal{A}$; since the right-hand side of $(23)_1$ is determined by the Navier-Stokes solution v_1 and the material coefficient $(\beta_2+\beta_3)/\mu_0$, the solution v_3 depends on those quantities. The biharmonic equation (24) has one and only one solution q_3 that satisfies the conditions $q_3 = 0$, $\partial_n q_3 = 0$ upon $\partial\mathcal{A}$. That solution is $q_3 = 0$. Hence $u_3 = 0$, and

$$\dot{\mathbf{x}} = [a\mathrm{v}_1(\mathrm{p}) + a^3\mathrm{v}_3(\mathbf{p})]\mathbf{k} + \mathbf{O}(a^4); \quad (6.4\text{-}25)$$

the function v_1 is determined uniquely by the cross-section and μ_0 and the function v_3 is determined uniquely by ρ and by $(\beta_2+\beta_3)/\mu_0$.

Step 6.4 (Approximate Solution For the Fluid of Grade 4). While the outcome of the step 2, given by (17), is an exact solution of the problem for the fluid of grade 2 if we simply drop the error term, the velocity field we have just obtained is only approximate for the general fluid of grade 3. Indeed, if we try to fit (25) with error term omitted to the theory of the grade 3 or grade 4 fluid, we cannot do so. Since the shear-viscosity function of such a fluid is given by $(6.1\text{-}22)_1$, we find that

$$\begin{aligned}\mu(\kappa)\nabla \mathrm{v}\\ &= \{\mu_0 + 2(\beta_2+\beta_3)[a^2\kappa_1^2 + 2a^4\nabla \mathrm{v}_1\cdot\nabla \mathrm{v}_3 + a^6|\nabla \mathrm{v}_3|^2]\}\,(a\nabla \mathrm{v}_1 + a^3\nabla \mathrm{v}_3)\\ &= \mu_0 a\nabla \mathrm{v}_1 + \mu_0 a^3\nabla \mathrm{v}_3 + 2(\beta_2+\beta_3)a^3\kappa_1^2\nabla \mathrm{v}_1 + O(a^5) \qquad (6.4\text{-}26)\end{aligned}$$

and that the error term $O(a^5)$ vanishes if and only if $\beta_2+\beta_3=0$—that is, if and only if $\mu=\mu_0$. Taking the divergence of (26), by use of $(23)_2$ and $(8)_1$ we see that

$$\operatorname{div}(\mu(\kappa)\nabla \mathrm{v}) = -a + O(a^5), \qquad (6.4\text{-}27)$$

and again the error term vanishes only for special values of v_1, which correspond with special cross-sections or with a fluid for which $\beta_2+\beta_3=0$. In order for the speed field v of a steady rectilinear flow to be possible, (5.7-10) must be satisfied. By (27), such is the case here, in general, only to within the error $O(a^5)$. Thus a *steady rectilinear flow is not strictly possible for the general fluid of grade 3 in a pipe of general cross-section.* In this sense the solution (25) that our formal but otherwise exact procedure delivers at step 3 is an approximate rather than an exact one. Of course from the very start we might have noticed that a steady rectilinear flow was possible for the particular fluids of grade 3 such that $\beta_2+\beta_3=0$, since for them the relation (5.7-13) is satisfied.

EXERCISE 6.4.4

Use of (23), (6.3-4), and (6.3-7) to show that

$$\begin{aligned}\mathbf{A}_1^2 &= a^2(\nabla \mathrm{v}_1\otimes\nabla \mathrm{v}_1 + \kappa_1^2\mathbf{k}\otimes\mathbf{k})\\ &\quad + a^4[\nabla \mathrm{v}_1\otimes\nabla \mathrm{v}_3 + \nabla \mathrm{v}_3\otimes\nabla \mathrm{v}_1 + 2(\nabla \mathrm{v}_1\cdot\nabla \mathrm{v}_3)(\mathbf{k}\otimes\mathbf{k})] + \mathbf{O}(a^5),\\ \mathbf{A}_2 &= 2a^2\nabla \mathrm{v}_1\otimes\nabla \mathrm{v}_1 + 2a^4(\nabla \mathrm{v}_1\otimes\nabla \mathrm{v}_3 + \nabla \mathrm{v}_3\otimes\nabla \mathrm{v}_1) + \mathbf{O}(a^5),\\ \mathbf{A}_2^2 &= 4a^4\kappa_1^2\nabla \mathrm{v}_1\otimes\nabla v_1 + \mathbf{O}(a^6),\\ \operatorname{tr}\mathbf{A}_2 &= 2a^2\kappa_1^2 + 4a^4\nabla \mathrm{v}_1\cdot\nabla \mathrm{v}_3 + \mathbf{O}(a^5),\\ \mathbf{A}_3 &= \mathbf{O}(a^5),\ \mathbf{A}_4 = \mathbf{O}(a^6). \qquad (6.4\text{-}28)\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{t} &= a(\ldots) + a^3(\ldots) + a^4\mu_0\nabla \mathrm{v}_4 + \mathbf{O}(a^5),\\ \mathbf{\Pi} &= a^2(\ldots) + a^4\Big\{\mu_0[\nabla\mathbf{u}_4 + (\nabla\mathbf{u}_4)^T]\\ &\quad + (2\alpha_1+\alpha_2)(\nabla \mathrm{v}_3\otimes\nabla \mathrm{v}_1 + \nabla \mathrm{v}_1\otimes\nabla \mathrm{v}_3)\\ &\quad + \gamma\kappa_1^2\nabla \mathrm{v}_1\otimes\nabla \mathrm{v}_1\Big\} + \mathbf{O}(a^5), \qquad (6.4\text{-}29)\end{aligned}$$

in which

$$\gamma := 4(\gamma_3+\gamma_4+\gamma_5+\frac{1}{2}\gamma_6). \qquad (6.4\text{-}30)$$

We are now ready to find the consequences of the dynamical equations (6.3-12) to order a^4. To this end we substitute (29) into the left-hand sides. Since we now know that $\mathbf{u} = \mathbf{O}(a^4)$, we see that

$$\mathbf{u} \cdot \nabla \mathrm{v} = O(a^5), (\nabla \mathbf{u})\mathbf{u} = \mathbf{O}(a^8), \tag{6.4-31}$$

so the right-hand sides of (6.3-12) vanish. Thus, even at step 4 the acceleration is negligible; therefore, the solution we are obtaining is one for "slow flow" in the sense that we could have neglected the acceleration from the start, had we but known it was justifiable to do so. (For the fluids of grades 1 and 2, as we have shown, the acceleration vanishes exactly.)

EXERCISE 6.4.5
Show that

$$\begin{aligned} &\mu_0 \Delta \mathrm{v}_4 = 0, \\ &\mu_0 \operatorname{div}[\nabla \mathbf{u}_4 + (\nabla \mathbf{u}_4)^T] + (2\alpha_1 + \alpha_2) \operatorname{div}[\nabla \mathrm{v}_3 \otimes \nabla \mathrm{v}_1 + \nabla \mathrm{v}_1 \otimes \nabla \mathrm{v}_3] \\ &\qquad + \lambda \operatorname{div}[\kappa_1^2 \nabla \mathrm{v}_1 \otimes \nabla \mathrm{v}_1] - \nabla \zeta_4 = 0. \end{aligned} \tag{6.4-32}$$

Hence $\mathrm{v}_4 = 0$, and $(32)_2$ simplifies to deliver

$$\mu_0 \Delta\Delta q_4 = \delta (\nabla \mathrm{v}_1)^{\perp} \cdot \nabla \operatorname{div}[|\nabla \mathrm{v}_1|^2 \nabla \mathrm{v}_1], \tag{6.4-33}$$

in which

$$\delta := \gamma - \frac{2}{\mu_0}(2\alpha_1 + \alpha_2)(\beta_2 + \beta_3). \tag{6.4-34}$$

For the fluid of grade 4 the condition $\delta = 0$ is necessary and sufficient that (15) be satisfied with q_4 replacing q_2 and hence that the fluid may undergo steady rectilinear flow for all a.

If $\delta \neq 0$, (33) is an inhomogeneous biharmonic equation for q_4, so it has a unique solution, determined by its right-hand side and hence by the Navier-Stokes speed v_1 and by the constant δ, which is a particular combination of nine of the fourteen viscosities of orders 1, 2, 3, and 4. Except for fluids such that $\delta = 0$ or for cross-sections such that $(\nabla \mathrm{v}_1)^{\perp} \cdot \nabla \operatorname{div}(|\nabla \mathrm{v}_1|^2 \nabla \mathrm{v}_1) = 0$, the solution q_4 of (33) will not vanish identically. Therefore, in general, a secondary flow appears at the fourth step in the formal expansion constructed here. The outcome, then, is

$$\dot{\mathbf{x}} = (a\mathrm{v}_1 + a^3 \mathrm{v}_3)\mathbf{k} + a^4 (\nabla q_4)^{\perp} + \mathbf{O}(a^5); \tag{6.4-35}$$

the functions v_1, v_3, and q_4 are determined uniquely by the cross-section $\mathcal{A}$ and by the viscosities of orders 1 through 4, and none of them vanishes except for special fluids or special cross-sections.

The process we have used delivers a unique solution at each step. For certain very special fluids, as we have seen, the process terminates after a certain finite number of steps and thus delivers an exact solution of the problem. While the process shows that the solution is the only one expressible as a polynomial in a, there remains a possibility that other solutions, solutions that are not such polynomials, may exist for the same boundary-value problem. Even for the Navier-Stokes fluid this remark is not an idle one, since many persons, relying upon the evidence of experiment, think that for sufficiently great values of the driving force a there should be infinitely many unsteady solutions in addition to the classical steady one. At present there is no mathematical proof of this conjecture. Still more difficult, naturally, is the problem for a general fluid, since there is then no reason to expect that the formal power series in a terminates, and we have no proof that it converges for any interval of a or represents in any way a solution of the problem.

Mathematical difficulties of the highest order are encountered in any attempt to study continuum mechanics precisely, and the wonder is not that so few problems have been solved convincingly, but that any at all have been solved.

6.5 Secondary Flow down a Straight Pipe: Discussion

As we remarked at the beginning of step 4, a secondary flow results even for the fluid of grade 3 if $\beta_2 + \beta_3 \neq 0$, since if $\gamma_1 = \gamma_2 \cdots = \gamma_6 = 0$, then (6.4-34) generally yields a nonzero value of δ and hence, for a general cross-section, the stream function q_4 also fails to be constant. The only exceptions are provided by fluids such that $2\alpha_1 + \alpha_2 = 0$ or $\beta_2 + \beta_3 = 0$. In the former, by $(6.1\text{-}22)_2$ we see that $\sigma_1 := 0$ for a fluid of grade 2; in the latter, by $(6.1\text{-}22)_{1,2}$ we see that $\mu = \mu_0$ and σ_1 is proportional to κ^2; in both cases the special relation (5.7-13) holds, so a rectilinear flow is possible. In general, nonetheless, (6.4-35) with $\mathbf{O}(a^5)$ and γ replaced by 0 does not provide an exact solution for the fluid of grade 3.

Returning to the general conclusions in Section 4, which hold to within the error $\mathbf{O}(a^5)$ in our formal series expansion, we see that despite the length of the calculation, the conclusions are simple. The first term, v_1, is the Navier-Stokes field of speed, which is uniquely determined by the Poisson equation $(6.4\text{-}8)_1$ and the boundary condition $\mathrm{v}_1 = 0$ on $\partial\mathcal{A}$. Once we have v_1, we easily determine v_3 by the Poisson equation $(6.4\text{-}23)_1$ with the boundary condition $\mathrm{v}_3 = 0$ on $\partial\mathcal{A}$. Nevertheless, if our interest lies in the secondary flow alone, we may pass directly to u_4, the stream function of which is obtained by solving the biharmonic equation (6.4-33) with boundary condition $q_4 = 0$, $\partial_n q_4 = 0$ on $\partial\mathcal{A}$.

Let $\bar{\mathrm{v}}_1$, $\bar{\mathrm{v}}_3$, and $\bar{\mathrm{q}}_4$ denote the solutions of the problem defined by (6.4-1) and (6.4-2) for the fluid such that $\mu_0 = 1$, $\beta_2 + \beta_3 = 1$, $\delta = 1$ in some system of units. The corresponding solution for the fluid with 14 arbitrary viscosities μ_0, α_1, α_2,

$\beta_1, \beta_2, \beta_3, Y_1, \ldots, Y_8$ are given by

$$\mathrm{v}_1 = \frac{1}{\mu_0}\bar{\mathrm{v}}_1, \mathrm{v}_3 = \frac{\beta_2 + \beta_3}{4\mu_0}\bar{\mathrm{v}}_3, q_4 = \frac{\delta}{5\mu_0}\bar{q}_4. \tag{6.5-1}$$

The curves $q_4 = \text{const.}$ are the streamlines of the secondary flow. Since these are the same as the curves $\bar{q}_4 = \text{const.}$, the secondary flow pattern is the same for all fluids in which $\mu_0\delta \neq 0$. As is clear from (6.4-35), the flow is generally helicoidal, making the fluid points travel down the tube in some sort of spirals. The pitch of these spirals, unlike the secondary pattern alone, varies from one fluid to another and depends also upon the specific driving force a. So also do the distribution of speeds along the streamlines and the tractions the pipe must exert in order to contain the fluid in its flow.

In Section 5.7 we saw that a steady rectilinear flow adhering to a pipe is a viscometric flow but generally is not dynamically possible unless some particular body force that is not lamellar is supplied. The foregoing analysis determines, to within specified assumptions and a remainder terms $\mathbf{O}(a^5)$, the flow that does occur in a fluid body of grade n when the specific driving force a alone is applied. This flow is not viscometric. Nevertheless, as the assertions in the following exercise show, it is completely determined by the viscometric functions of the fluid, providing that μ be twice differentiable and σ_1 four times differentiable.

EXERCISE 6.5.1

Show that the truncated viscometric functions (6.1-22) of the fluid of grade 4 show that the perturbation v_3 of the longitudinal speed is determined for any given cross-section by the quantity

$$\frac{\mu''(0)}{2\mu_o^4}, \tag{6.5-2}$$

while the quantity δ determining the secondary flow is given by

$$\delta = \frac{\sigma_1''(0)}{24} - \frac{\sigma_1''(0)\mu''(0)}{4\mu_o}. \tag{6.5-3}$$

These conclusions have a double importance. First, the flow pattern is more ebullient than that predicted by the Navier-Stokes theory. Second, information gleaned in viscometric flows suffices to predict the flow that does occur, though it is not viscometric. Of course, for the Navier-Stokes fluid and the fluid of second grade, *all* flows are determined by viscometric information, and originally the aim of viscometry was to ascertain the nature of a particular fluid and hence to determine, in principle, all flows that bodies of that fluid might experience. The solution for secondary flow in pipes shows that data obtained by experiments on viscometric flows may serve in principle to predict the nature of a phenomenon that occurs in a flow that is not viscometric. We notice that the function σ, which has no effect on the possibility or impossibility of rectilinear flow, also has no effect on the approximate flow according to the theory of the fluid of grade 3 or more.

Since the theory of the partial differential equations found to occur at the successive stages is standard, we may (with some reservation) regard the problem of calculating secondary flows according to the theory of the fluid of grade n as solved to within an error $\mathbf{O}(a^5)$ by the foregoing analysis. The conclusions may be illustrated by the example of the elliptical pipe. The Navier-Stokes solution has been obtained already as (5.7-15), whence it follows that

$$\mathrm{v}_1 = -\frac{c^2 b^2}{2\mu_0(c^2+b^2)}\left(\frac{x_1^2}{c^2}+\frac{x_2^2}{b^2}-1\right). \tag{6.5-4}$$

EXERCISE 6.5.2 (Green and Rivlin)
Show that for the elliptical pipe (6.4-33) becomes

$$\mu_0^5 \Delta\Delta q_4 = \delta \frac{6c^2b^2(c^2-b^2)}{(c^2+b^2)^3} x_1 x_2. \tag{6.5-5}$$

Hence the stream function of the secondary flow is given by

$$q_4 = \frac{\delta}{\mu_o^5} A \left(\frac{x_1^2}{c^2}+\frac{x_2^2}{b^2}-1\right) x_1 x_2,$$

$$A = \frac{c^6 b^6 (c^2-b^2)}{4(c^2+b^2)(5c^4+6c^2b^2+5b^4)}. \tag{6.5-6}$$

Here we have discussed the presence of secondary flows due to a small driving force. However, it is possible to study the possibility of secondary flows when the driving force is not small, but the departure from the circularity of the cross-section is small. The recent study of Mollica and Rajagopal[6] discusses this problem in great detail.

It is often said that the first few terms of a power series rarely enlighten because, since their sum does not stray far from what was known before, they are fit to describe only situations essentially classical. The problem of secondary flow in a straight pipe affords one of the rare cases in which a power series delivers something not routine. As $a \to 0$, the resulting velocity field approaches the classical one, it is true. Nonetheless, the presence of any steady component, however small, of velocity normal to the main flow yields helicoidal streamlines. Therefore, the fluid points are not caused to remain arbitrarily close to their classical positions by

[6] F. Mollica and K.R. Rajagopal, "Secondary flows due to axial shearing of a third grade fluid between two eccentrically placed cylinders," *International Journal of Engineering Science* 37, 411–419 (1999).

making a arbitrarily small. True, the smaller a is, the longer the pipe will have to be in order to reveal the difference from the classical prediction. But should we peer down the infinite pipe, we should see (with eyesight of infinite range) the projections of the streamlines as closed curves whose forms are independent of a.

6.6 Universal Flows of Fluids of Grades 1, 2, and 3

Here we shall consider only fluids subject to lamellar body force. A flow universal for fluids of grade n may be universal for fluids of grade $n+1$, but it need not be. Of course a flow universal for fluids of grade n is universal for fluids of grade less than n. For example, all flows of an Eulerian fluid are universal; they are those that preserve circulation. Therefore, since Eulerian fluids are particular fluids of grade n for every n, *all universal flows of fluids of grade n preserve circulation:*

$$\operatorname{skw}\operatorname{grad}\ddot{\mathbf{x}} = \mathbf{0}, \tag{2.2-2}$$

$$\ddot{\mathbf{x}} = -\operatorname{grad} P_a. \tag{2.2-9}$$

For a fluid whose determinate response is $\mathcal{G}$ to undergo a flow that preserves circulation, it is necessary and sufficient that if $\mathbf{F}^t$ gives rise to that flow, then

$$\operatorname{skw}\operatorname{grad}\operatorname{div}\mathcal{G}(\mathbf{F}^t) = \mathbf{0}, \tag{3.8-6}$$

and hence there is a scalar field λ such that

$$\operatorname{div}\mathcal{G}(\mathbf{F}^t) = -\operatorname{grad}\lambda, \tag{3.8-7}$$

and the pressure is given by

$$p = \rho(P_a - \varpi) - \lambda. \tag{3.8-8}$$

Taking ϖ as 0 reminds us that *universal flows can be produced by applying surface tractions alone.* Conversely, nonetheless, surface tractions do not suffice to make an arbitrary fluid undergo an arbitrary flow that preserves circulation. Moreover, (3.8-8) shows us that *once a flow is known to be universal for a certain fluid, the pressure required to effect it is a linear combination of three potentials.*

We now address the problem of finding flows of fluids of grades 1, 2, and 3 that are universal for them but are not universal for all fluids.

6.6.1 Navier-Stokes fluid

For the Navier-Stokes theory, $n = 1$, and the constitutive equation for the determinate stress is

$$\mathcal{G}(\mathbf{F}^t) = \mu\mathbf{A}_1, \quad \mu > 0, \tag{6.6-1}$$

so (3.8-6) reduces to

$$\text{skw grad div } \mathbf{A}_1 = \mathbf{0}, \tag{6.6-2}$$

Thus there must be a potential P_f such that

$$\text{div } \mathbf{A}_1 - 2 \text{ grad } P_f, \tag{6.6-3}$$

and in (3.8-8), $\lambda = 2\mu P_f$ for the Navier-Stokes fluid. Application of (3.8-6) gives the required pressure field:

$$\varphi = P_a - 2\mu P_f \tag{6.6-4}$$

The condition (3) is familiar in classical studies of the kinematics of fluids. Because div $\mathbf{A}_1 = 2$ div $\mathbf{W}$ in an isochoric flow, (3) states that div $\mathbf{W} = -$ grad P_f, so P_f is called a *flexion potential*. Because div div $\mathbf{W} = 0$, necessarily $\Delta P_f = 0$, and various interesting consequences follow.[7]

In summary, we may obtain all Navier-Stokes universal flows as follows.

1. Determine every Euler flow that satisfies (2).
2. Exhibit, by use of (4), the pressures required to effect them.

Steady universal solutions of Navier-Stokes equation are particularly suited to determining μ by experiment. We assign ϖ to within an arbitrary constant h, determine h from boundary conditions, and calculate both P_a and P_f from the given flow $\dot{\mathbf{x}}$, known to be universal. We then measure p at some one point. Since μ is a constant, that amount of data allows us to calculate μ from (4). Examination of some procedures used to determine μ by experiment shows that in principle they rest upon this idea. An example is provided by the steady shearings,

$$\dot{\mathbf{x}} = \text{v}(\mathbf{p})\mathbf{k}, \tag{5.7-1}$$

where $\mathbf{p}$ is a position vector in a plane normal to the unit vector $\mathbf{k}$. These flows are isochoric and steady; in Section 5.7 we saw that they are viscometric and that they satisfy the dynamical equation if and only if

$$\Delta \text{v} = -C = \text{const.} \tag{6.6-5}$$

(see the text following (5.7-13)).

EXERCISE 6.6.1

The condition (5) is necessary and sufficient for (3) to be satisfied, and if so

$$2P_f = Cz, \tag{6.6-6}$$

where z is a coordinate on a line parallel to $\mathbf{k}$.

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[7]These and other developments are due to Thomas Craig. See pp. 44 and 48 of C. Truesdell, *The Kinematics of Vorticity* (Bloomington: Indiana University Press, 1954).

There are further universal flows, as can easily be verified by specializing suitably the viscometric flows analyzed in Section 5.4. To delimit and display the whole class of universal solutions is quite another problem. It has resisted the efforts of expert hydrodynamicists for more than a century. Recently Marris, Yin, and others have had noteworthy success in finding some universal solutions and excluding some putative classes.

A screw flow is a rotational flow in which values of the velocity field and the spin field are parallel at each point and time. For a steady flow of this kind, (2.2-29) delivers an acceleration potential: $\ddot{\mathbf{x}} = \operatorname{grad}(\frac{1}{2}\dot{\mathbf{x}}^2)$. To be a universal Navier-Stokes flow, the velocity field would have to satisfy (2). In a difficult, complicated analysis Marris[8] has proved that in three dimensions the system

$$\operatorname{div}\dot{\mathbf{x}} = 0, \tag{1.4-6}$$

$$\mathbf{W}\dot{\mathbf{x}} = \mathbf{0}, \tag{6.6-7}$$

$$\operatorname{skw}\operatorname{grad}\operatorname{div}\mathbf{W} = \mathbf{0} \tag{6.6-8}$$

has no solution except $\mathbf{W} = \mathbf{0}$; *no steady screw flow is a Navier-Stokes flow.* Many studies of steady screw flows are rendered vacuous by this theorem.

A *complex-lamellar* flow is a rotational flow in which the axis of spin at each point is normal to the velocity there. Every plane rotational flow is complex-lamellar; so is every rotational axisymmetric flow. Marris and Ames[9] have determined and classified the vorticities of all steady, rotational, universal complex-lamellar Navier-Stokes flows:

1. Plane or axisymmetric flows

2. Flows whose streamlines are parallel straight lines

3. Flows obtained by superposing a steady, irrotational flow on the circular helical flow of Strakhovitch, namely, in cylindrical physical components

$$\dot{x}_r = 0, \quad \dot{x}_\theta = \frac{d}{r} + \frac{kar}{2}, \quad \dot{x}_z = -\frac{r}{a}\dot{x}_\theta = -\frac{d}{a} - \frac{kr^2}{2}, \tag{6.6-9}$$

where d, k, and a are constants. An example belonging to Marris and Ames's third class is seen in the following exercise[10]

[8] A.W. Marris, "The impossibility of steady screw motions of a Navier-Stokes fluid," *Archive for Rational Mechanics and Analysis* 70 (1979): 47–66.

[9] A.W. Marris and W.F. Ames, "Addendum: on complex-lamellar motions," *Archive for Rational Mechanics and Analysis* 64 (1977): 371–79. The conclusions in this paper rest in part on an earlier work by Marris, "On complex-lamellar motions," ibid. 59 (1975): 131–38. See also A.W. Marris and M.P. Stallybrass, "A class of fluid motions obtained by superposition," *Atti della Accademia di Scienze di Torino* 113 (1979): 53–60.

[10] A general representation of the velocity fields is given by M.P. Stallybrass and A.W. Marris, "Some new exact solutions of the Navier-Stokes equations associated with steady vortex flows," *Letters in Applied Engineering Science* 5 (1977): 359–66.

EXERCISE 6.6.2

Show that the isochoric flow

$$\begin{aligned} \dot{x}_r &= -\frac{c}{a} \cdot \frac{z-a}{r} + \frac{b}{r}, \\ \dot{x}_\theta &= c\frac{\log r}{r} + \frac{d}{r} + \frac{kar}{2}, \\ \dot{x}_z = -\frac{r}{a}\dot{x}_\theta &= -\frac{c}{a}\log r - \frac{d}{a} - \frac{kr^2}{2}, \end{aligned} \qquad (6.6\text{-}10)$$

where a, b, c, d, and k are constants, is a complex-lamellar, universal Navier-Stokes flow.

Even plane flows are troublesome. By (2.5-11) they may be described in terms of a stream function q, which here must solve the system

$$\partial_t \Delta q + (\nabla q)^T \cdot \nabla \Delta q = 0, \quad \Delta\Delta q = 0, \qquad (6.6\text{-}11)$$

where ∇ and Δ are the gradient operator and the Laplacian operator in the plane. The first of these equations makes the flow have an acceleration potential, the second, a flexion potential. (If we look back at (6.4-8) and (6.4-11), we see that in the series expansion presented in Section 6.4 the first term is provided by a universal Navier-Stokes flow.)

W.-L Yin,[11] broadly extending and completing an earlier analysis by Kampe De Feriet and others, has determined and classified all plane, universal Navier-Stokes flows, steady or not. His analysis, which employs functions of a complex variable, is ingenious and difficult.

For axisymmetric isochoric flows to preserve circulation, it is necessary and sufficient that the ratio of vorticity to the distance from the axis remain constant for each fluid point:[12]:

$$w/r = K, \ \dot{K} = 0. \qquad (6.6\text{-}12)$$

Marris and Aswani[13] have proved that all steady universal Navier-Stokes flows that are axisymmetric, other than those whose streamlines are parallel straight lines, must satisfy (13) with K constant in space.

From all the foregoing we may conclude that even the relatively few universal Navier-Stokes flows are too numerous to survey succinctly. Formally, the general solution of the system constituted by (2.2-27) and (2) remains unknown.

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[11] W.-L. Yin, "Circulation-preserving plane flows of incompressible viscous fluids," *Archive for Rational Mechanics and Analysis* 83 (1983): 169–84.

[12] Svanberg's theorem, CFT 133 C. Truesdell and R. Toupin, Classical Field Theory, *Handbuch der Physik*, 3 (Springer-Verlag: Berlin, Göttingen, and Heidelberg, 1960).

[13] A.W. Marris and M.G. Aswani, "On the general impossibility of controllable axisymmetric Navier-Stokes motions," *Archive for Rational Mechanics and Analysis* 63 (1977): 107–53.

6.6.2 Fluids of Grade 2

To study universal flows of fluids of grades greater than 1, we follow the treatment of Fosdick and Truesdell. We should expect the universal flows of fluids of grade 2 to make up a proper subclass of the universal Navier-Stokes flows. A glance at (6.1-17) shows that now

$$\mathcal{G}(\mathbf{F}^t) = \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{6.6-13}$$

in which μ, α_1, and α_2 are constants. Universal flows must make the right-hand side equal to a gradient. Because of (3), $\operatorname{div}(\alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2)$ must equal a gradient. Giesekus showed that if (3) holds, then for isochoric flows

$$\operatorname{div}(\mathbf{A}_2 - \mathbf{A}_1^2) = -\operatorname{grad}\left(2\dot{P}_f - \frac{1}{4}\operatorname{tr}\mathbf{A}_1^2\right). \tag{6.6-14}$$

We shall establish this identity a little further on. Use of it shows that

$$-\alpha \ \operatorname{grad}\left(2\dot{P}_f - \frac{1}{4}\operatorname{tr}\mathbf{A}_1^2\right) = (\alpha_1 + \alpha_2)\operatorname{div}\mathbf{A}_2, \tag{6.6-15}$$

so Navier's dynamical equation reduces to

$$-\operatorname{grad}\left[P_{\mathfrak{a}} - \varphi - \mathrm{v}P_{\mathfrak{f}} + \alpha_2\left(\dot{P}_{\mathfrak{f}} - \frac{1}{4}\operatorname{tr}\mathbf{A}_1^2\right)\right] = (\alpha_1 + \alpha_2)\operatorname{div}\mathbf{A}_2. \tag{6.6-16}$$

Thence follows the necessary and sufficient condition

$$(\alpha_1 + \alpha_2)\operatorname{div}\mathbf{A}_2 = -(\alpha_1 + \alpha_2)\operatorname{grad} P_2, \tag{6.6-17}$$

in which P_2 is a scalar field. Conversely, if this condition as well as (2.11-47) and (3) are satisfied, then Navier's dynamical equation is satisfied, and the pressure required to effect the corresponding universal flow is given by

$$\varphi = P_{\mathfrak{a}} - 2\mu P_{\mathfrak{f}} + \alpha_2\left(2\dot{P}_{\mathfrak{f}} - \frac{1}{4}\operatorname{tr}\mathbf{A}_1^2\right) = (\alpha_1 + \alpha_2)\operatorname{div}\mathbf{A}_2. \tag{6.6-18}$$

Hence *the fluids of grade 2 such that* $\alpha_1 + \alpha_2 = 0$ *have just the same universal flows as the Navier-Stokes fluids; those for which* $\alpha_1 + \alpha_2 \neq 0$ *have as their universal flows only those that satisfy the additional condition*

$$\operatorname{div}\mathbf{A}_2 = -\operatorname{grad} P_2. \tag{6.6-19}$$

For an example we return to the steady shearings (5.7-1).

EXERCISE 6.6.3

Show that the steady shearings satisfy (19) if they satisfy (5), and

$$P_2 = 2Cv - |\nabla v|^2,$$

$$\begin{aligned} \dot{P}_{\mathfrak{f}} &= Cv, \\ \operatorname{tr} \mathbf{A}_1^2 &= 2|\nabla v|^2, \\ \varphi &= -\mu C x - (2\alpha_1 + \alpha_2)\left(Cv - \frac{1}{2}|\nabla v|^2\right). \end{aligned} \tag{6.6-20}$$

Thus the steady universal flows for a Navier-Stokes fluid are universal for fluids of grade 2, but a pressure proportional to $Cv - \frac{1}{2}|\nabla \mathrm{v}|^2$ must be added to or subtracted from the Navier-Stokes pressure in order to effect them.

As the second example we turn to plane flows and show that the *class of plane universal flows for fluids of grade 2 is the same as that for Navier-Stokes fluids.*

EXERCISE 6.6.4
Show that

$$\operatorname{div} \mathbf{A}_1^2 = \nabla[2|\nabla\nabla q|^2 - (\Delta q)^2] = \frac{1}{2}\nabla \operatorname{tr} \mathbf{A}_1^2. \tag{6.6-21}$$

Thus (15) and (12) imply that (19) holds,

$$P_2 = 2\dot{P}_{\mathfrak{f}} - \frac{3}{2}[2|\nabla\nabla q|^2 - (\Delta q)^2], \tag{6.6-22}$$

and

$$\varphi = -2P_{\mathfrak{f}} - 2\alpha_1 \dot{P}_{\mathfrak{f}} + \frac{1}{2}(3\alpha_1 + 2\alpha_2)[2|\nabla\nabla q|^2 - (\Delta q)^2]. \tag{6.6-23}$$

6.6.3 *Fluids of Grade 3*

A glance at (6.1-19)$_2$ shows that now

$$\mathcal{G}(\mathbf{F}^t) = \ldots \beta_1 \mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3[(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_1], \tag{6.6-24}$$

in which the dots stand for the right-hand side of (14). Following the pattern of the treatment of the fluid of grade 2, we now see that for a universal flow, the divergence of the right-hand side of (24) must equal a gradient. A little further on we shall prove an identity that delivers as a special instance not only (15) but also the following:

$$\begin{aligned} &\operatorname{div}[(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_1 - (\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1)] \\ &= -\operatorname{grad}\left(2\ddot{P}_{\mathfrak{f}} - \dot{P}_2 - \tfrac{1}{4}\,\dot{\overline{\operatorname{tr}\mathbf{A}^2}} + \tfrac{5}{6}\operatorname{tr}\mathbf{A}_1^3\right), \end{aligned} \tag{6.6-25}$$

on the assumption that $\operatorname{div}\dot{\mathbf{x}} = 0$ and that (3) and (19) hold. Therefore Navier's equation reduces to

$$\begin{aligned} -\operatorname{grad}\Big[\rho(P_{\mathrm{a}} - \varpi) - p - 2\mu P_{\mathrm{f}} + \alpha_2\left(2\dot{P}_{\mathrm{f}} - \frac{1}{4}\operatorname{tr}\mathbf{A}_1^2\right) - (\alpha_1 + \alpha_2)P_2 \\ + \beta_2\left(2\ddot{P}_{\mathrm{f}} - \dot{P}_2 - \frac{1}{4}\,\overline{\operatorname{tr}\mathbf{A}_1^2}^{\,\cdot} + \frac{5}{6}tr\mathbf{A}_1^3\right) - p/\rho\Big] \\ = \beta_1\operatorname{div}\mathbf{A}_3 + (\beta_2 + \beta_3)\operatorname{div}[(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_1]. \end{aligned} \qquad (6.6\text{-}26)$$

Thus for the universal flows of fluids of grade 3, not only $\ddot{\mathbf{x}}$, $\mu\mathbf{A}_1$, and $(\alpha_1 + \alpha_2)\mathbf{A}_2$ must equal gradients but so also must

$$\beta_1\operatorname{div}\mathbf{A}_3 \qquad \text{and} \qquad (\beta_2 + \beta_3)\operatorname{div}[(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_1]. \qquad (6.6\text{-}27)$$

To complete the arguments given in treating fluids of grades 2 and 3, we need to establish the identities (25) and (15). Both of these are consequences of the following Fundamental identity (Fosdick and Truesdell.[14] *Let* $\mathbf{v}$ *be a velocity field, from which* $\mathbf{A}_1$ *derives; let* $\mathbf{A}$ *be any symmetric tensor field deriving from a potential,*

$$\operatorname{div}\mathbf{A} = -\operatorname{grad}\varphi, \qquad (6.6\text{-}28)$$

and let $\mathbf{B}$ be defined as follows:

$$\mathbf{B} := \dot{\mathbf{A}} + \mathbf{A}\operatorname{grad}\mathbf{v} + (\mathbf{A}\operatorname{grad}\mathbf{v})^T. \qquad (6.6\text{-}29)$$

Then

$$\operatorname{grad}\varphi = -\operatorname{div}\mathbf{B} + \frac{1}{2}(\operatorname{grad}\mathbf{A}_1) + \operatorname{div}(\mathbf{A}\mathbf{A}_1). \qquad (6.6\text{-}30)$$

This identity deserves to be called fundamental because it can be used to reduce conditions of integrability for fluids of grade higher than 3.

PROOF First we differentiate (28) substantially :

$$\operatorname{grad}\dot{\varphi} = -\operatorname{div}\dot{\mathbf{A}} + \operatorname{div}[(\mathbf{gradA})\mathbf{v}] - \operatorname{grad}[(\operatorname{div}\mathbf{A})\cdot\mathbf{v}]. \qquad (6.6\text{-}31)$$

Insertion of (29) here yields

$$\begin{aligned} \operatorname{grad}\dot{\varphi} = &-\operatorname{div}\mathbf{B} + \operatorname{div}[\mathbf{A}\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^T\mathbf{A} + (\operatorname{grad}\mathbf{A})\mathbf{v}] \\ &- \operatorname{grad}[(\operatorname{div}\mathbf{A})\cdot\mathbf{v}], \end{aligned} \qquad (6.6\text{-}32)$$

[14]R.L. Fosdick and C. Truesdell, "Universal flows in the simplest theories of fluids," *Annali della Scuola Normale Superiore di Pisa* 4 (1977): 323–41.

EXERCISE 6.6.5
Show that for a field $\mathbf{v}$ and a symmetric tensor $\mathbf{A}$, sufficiently differentiable,

$$\operatorname{grad}[(\operatorname{div}\mathbf{A})\cdot\mathbf{v}] = (\operatorname{grad}\operatorname{div}\mathbf{A})^T\mathbf{v} + (\operatorname{grad}\mathbf{v})^T\operatorname{div}\mathbf{A},$$
$$\operatorname{div}[(\operatorname{grad}\mathbf{v})^T\mathbf{A}] = (\operatorname{grad}\mathbf{v})^T\operatorname{div}\mathbf{A} + \mathbf{A}\cdot[\operatorname{grad}(\operatorname{grad}\mathbf{v})^T], \qquad (6.6\text{-}33)$$
$$\operatorname{div}[(\operatorname{grad}\mathbf{A})\mathbf{v}] = (\operatorname{grad}\operatorname{div}\mathbf{A})\mathbf{v} + \operatorname{div}[\mathbf{A}(\operatorname{grad}\mathbf{v})^T]. \qquad (6.6\text{-}34)$$

Application of these identities to (32) completes the proof.

To establish (15), we take $\mathbf{A}_1$ for $\mathbf{A}$ in (30) and recall that $(\operatorname{grad}\mathbf{A}_1)\mathbf{A}_1 = \frac{1}{2}\operatorname{grad}\operatorname{tr}\mathbf{A}_1^2$. Then (29) shows that $\mathbf{B} = \mathbf{A}_2$, and thence $\varphi = 2P_{\mathfrak{f}}$ by comparison of (26) with (3).

To establish (25), we take $\mathbf{A}_1^2$ for $\mathbf{A}$ in (30). Then (28) is satisfied, since by (15) and (18)

$$\operatorname{div}\mathbf{A}_1^2 = -\operatorname{grad}\left(P_3' - 2P_{\mathfrak{f}} + \frac{1}{4}\operatorname{tr}\mathbf{A}_1^2\right). \qquad (6.6\text{-}35)$$

Thus in (28) $\varphi = P_3 - 2P_{\mathfrak{f}} + \frac{1}{4}\operatorname{tr}\mathbf{A}_1^2$, so (30) delivers

$$\operatorname{grad}\left(\dot{P}_3 - 2\ddot{P}_{\mathfrak{f}} + \frac{1}{4}\,\overline{\operatorname{tr}\mathbf{A}_1^2}^{\,\cdot}\right) = -\operatorname{div}(\mathbf{B} - \mathbf{A}_1^3) + \frac{1}{6}\operatorname{grad}(\operatorname{tr}\mathbf{A}_1^3). \qquad (6.6\text{-}36)$$

Here $\mathbf{B}$ is given by (29) as

$$\mathbf{B} = \overline{\mathbf{A}_1^2}^{\,\cdot} + \mathbf{A}_1^2\operatorname{grad}\mathbf{v} + (\mathbf{A}_1^2\operatorname{grad}\mathbf{v})^T, \qquad (6.6\text{-}37)$$

or, equivalently,

$$\mathbf{B} = \overline{\mathbf{A}_1^2}^{\,\cdot} + \mathbf{A}_1^3 + \mathbf{A}_1^2\mathbf{W} - \mathbf{W}\mathbf{A}_1^2. \qquad (6.6\text{-}38)$$

The second Rivlin-Ericksen tensor may also be written in the form

$$\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1^3 + \mathbf{A}_1\mathbf{W} - \mathbf{W}\mathbf{A}_1, \qquad (6.6\text{-}39)$$

so

$$\mathbf{B} - \mathbf{A}_1^3 = \mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1 - 2\mathbf{A}_1^3. \qquad (6.6\text{-}40)$$

From the Hamilton-Cayley equation we know that

$$\mathbf{A}_1^3 = \frac{1}{2}(\operatorname{tr}\mathbf{A}_1^2)\mathbf{A}_1 + \frac{1}{3}(\operatorname{tr}\mathbf{A}_1^3)\mathbf{1}. \qquad (6.6\text{-}41)$$

Recalling that $\operatorname{tr}\mathbf{A}_2 = \operatorname{tr}\mathbf{A}_1^2$, by use of (39) we obtain

$$\mathbf{B} - \mathbf{A}_1^3 = \mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1 - (\operatorname{tr}\mathbf{A}_2)\mathbf{A}_1 - \frac{2}{3}(\operatorname{tr}\mathbf{A}_1^3)\mathbf{1}. \qquad (6.6\text{-}42)$$

Applying this formula to (35) yields (25), which led to the necessary and sufficient conditions for universal solutions of a fluid of grade 2, namely

$$\beta_1\operatorname{skw}\operatorname{grad}\operatorname{div}\mathbf{A}_3 = 0,$$

$$(\beta_2 + \beta_3)\,\text{skw}\,\text{grad}\,\text{div}[(\text{tr}\,\mathbf{A}_2)\mathbf{A}_1] = 0. \tag{6.6-43}$$

6.6.4 *Formal summary*

If we set aside special values for the coefficients $\alpha_1 + \alpha_2$, β_1, and $\beta_2 + \beta_3$, necessary and sufficient conditions for universal solutions of grades 0, 1, 2, and 3 are

$$0 = \text{skw}\,\text{grad}\begin{cases} \ddot{\mathbf{x}} & \text{grades } 0, 1, 2, \ldots \\ \text{div}\,\mathbf{A}_1 & \text{grades } 1, 2, \ldots \\ \text{div}\,\mathbf{A}_2 & \text{grades } 2, 3, \ldots \\ \text{div}[(\text{tr}\,\mathbf{A}_2)\mathbf{A}_1] & \text{grades } 3, 4, \ldots \\ \text{div}\,\mathbf{A}_3 & \text{grades } 3, 4, \ldots. \end{cases} \tag{6.6-44}$$

These suggest the beginnings of an iterative scheme for obtaining the classes of universal flows for fluids of grade n. The system might suggest that for a fluid of sufficiently high grade there would be no universal solutions, but we know that is not true, for, as we have seen, there are flows universal for the whole class of simple fluids—indeed, there are even universal viscometric flows.

6.6.5 *An approach to determining all universal flows*

The larger the constitutive class considered, the fewer the universal solutions. To show that a certain flow is not universal for all simple fluids, we need only show that it is not universal in some particular subclass. At present the determination of all universal flows for homoegenous, incompressible simple fluids is an open problem. We could attempt to solve it by determining the entire class of universal flows for fluids of some low grade, such as 3 or 4. If we could exhibit these flows, it would then be a simple matter to see if they were indeed universal for all simple fluids. For discovery, this approach is more systematic than the method of trial and error, which has been used to find such universal solutions as are now known, and it is mathematically easier than would be a frontal attack upon the equations of motion of a simple fluid in general. Maybe at grade 3 or grade 4 all universal flows will be found.

This approach has been illustrated by carrying it through for the steady shearings $(5.7\text{-}1)_1$. It is known that a shearing is universal for homegenous simple fluids if and only if $\mathbf{v}$ reduces to an affine function of two Cartesian coordinates y and z or of Arc tan (y/z). In fact these shearings constitute also the complete set of universal steady shearings for the fluid of grade 3.

The steady shearings are viscometric flows, so for them $\mathbf{A}_3 = \mathbf{0}$. Thus $(44)_5$ is satisfied, and $(44)_4$ becomes

$$\nabla \cdot (|\nabla \text{v}|^2 \nabla \text{v}) = K = \text{const.} \tag{6.6-45}$$

By complicated analysis, Fosdick and Truesdell proved that the only solutions to this equation are the simple shearings and the shearings of fanned planes. Thus the fluid of grade 3 has as its universal steady shearings precisely those that are universal for all fluids, no more. In other words, as far as steady shearings are concerned, in order to exhibit all universal flows it suffices to find the universal flows for a fluid of grade 3.

General Reference

[1.] Truesdell, C. and Noll, W., "*The Non-Linear Field Theories of Mechanics, Handbuch der Physik* 3_3." Berlin, Heidelberg, and New York: Springer-Verlag, 1965.

7

Some Flows of Fluids of Grade 2

7.1 Fluids of Grade 2

The incompressible fluid of grade 2 is defined by the constitutive relation (6.1-17), in which μ, α_1, and α_2 are constants, and μ, the shear viscosity, is positive. The sign of the coefficient α_1 has important effects on the nature of the solutions. If the constitutive relation is taken as defining a particular fluid, just as the Navier-Stokes fluid is almost always regarded, other restrictions (for example, those implied by thermodynamics) lead to the conclusion that $\alpha_1 \geq 0$, $\alpha_1 + \alpha_2 = 0$. However, strong sentiments have been espoused for assuming that $\alpha_1 < 0$ when this constitutive relation is regarded as a second-order approximation in the sense of retarded motions (6.1-9). A critical discussion of the relevant issues can be found in the recent review article by Dunn and Rajagopal.[1] In the following purely mechanical treatment we do not impose any of these adscititious inequalities. We instead emphasize the effects that the sign of α_1 has upon the phenomena associated with fluids of grade 2.

We note that $\mu/|\alpha_1|$ has the dimension of time. Thus the fluid of grade 2 may be expected to show evidence of having a time scale proper to itself. The special flows we study now will reveal some effects of the existence of this proper time. Of course these effects must vanish when $\alpha_1 = \alpha_2 = 0$, for then the fluid of grade 2 reduces to the Navier-Stokes fluid, with which no constitutive parameter bearing the dimension of time can be associated.

[1] J.E. Dunn and K.R. Rajagopal, "Fluids of differential type: critical review and thermodynamic analysis," *International Journal of Engineering Science* 33 (1995): 689–729.

7.2 Unsteady Lineal Flows

We consider again the velocity field of a time-dependent shearing:

$$\dot{x}_1 = 0, \ \dot{x}_2 = v(x_1, t), \ \dot{x}_3 = 0. \tag{5.5-1}$$

In Section 5.5 we obtained the general form of the components of stress for this flow, but here we shall solve directly the dynamical equation that arises when the fluid undergoing the flow is of the second grade. Substituting this velocity field into (6.1-17) and then substituting the outcome into Cauchy's first law (5.4-1), we obtain the following linear differential equation for v, in which we write x for x_1:

$$\mu\partial_x^2 v + \alpha_1\partial_t\partial_x^2 v + c(t) = \rho\partial_t v. \tag{7.2-1}$$

EXERCISE 7.2.1 (Coleman and Noll, Ting)
Verify that the value of $c(t)$ of c is the specific driving force.

The coefficient α_2 does not appear in (1). If $\alpha_1 = 0$, (1) reduces to the heat equation with source c, just as it does for the Navier-Stokes fluid, in the theory of which the lineal flow (5.5-1) occupies a prominent position as one of the few flows for which explicit solutions are easy to get. These solutions are important for illustrating the birth, growth, and decay of plane boundary layers. For a fluid of grade 2 such that $\alpha_1 \neq 0$, the field of speeds v differs from that appropriate to the Navier-Stokes equations. More than that, the partial differential equation to be solved is of third order rather than second. As might be expected, there is a greater range of possible solutions. The following text and exercises illustrate that fact. We shall suppose always that $c := 0$.

Markovitz and Coleman remarked that a particular solution of (1) is given by

$$v(x, t) = Ve^{-ax}\cos(\omega t - bx), \tag{7.2-2}$$

in which V and ω are assigned positive constants and

$$a := \sqrt{\frac{\rho\omega}{2\mu}\left(\frac{1}{\sqrt{1+\xi^2}} + \frac{\xi}{1+\xi^2}\right)} > 0, \tag{7.2-3}$$

$$b := \sqrt{\frac{\rho\omega}{2\mu}\left(\frac{1}{\sqrt{1+\xi^2}} - \frac{\xi}{1+\xi^2}\right)} > 0. \tag{7.2-4}$$

Here $\xi := \alpha_1\omega/\mu$, so $\operatorname{sgn}\xi = \operatorname{sgn}\alpha_1$; if $\alpha_1 \neq 0$, equations (3) and (4) can be given the equivalent forms

$$a = \sqrt{\frac{\rho\xi}{2\alpha_1}\left(\frac{1}{\sqrt{1+\xi^2}} + \frac{\xi}{1+\xi^2}\right)},$$

$$\mathrm{b} = \sqrt{\frac{\rho\xi}{2\alpha_1}\left(\frac{1}{\sqrt{1+\xi^2}} - \frac{\xi}{1+\xi^2}\right)}. \tag{7.2-5}$$

This solution may be regarded as representing a standing harmonic wave that an infinite oscillating plate $x = 0$ induces upon a fluid body adhering to it and filling the half-space $x > 0$. For the Navier-Stokes fluid $\alpha_1 = 0$, so

$$\mathrm{a} = \mathrm{b} = \sqrt{\frac{\rho\omega}{2\mu}}; \tag{7.2-6}$$

the absorption coefficient a and the phase shift b are equal, and both are proportional to $\sqrt{\omega}$, so for a given fluid the higher the frequency of the oscillation, the more the waves are absorbed at great distances from the driving plate. If $\alpha_1 \neq 0$, the nature of the shearing and the spin is quite different.

EXERCISE 7.2.2 (Truesdell)
When V and ω are assigned, show that the maximum shearing and vorticity are given by

$$\begin{aligned} K_{\text{max}}(\omega) = w_{\text{max}}(\omega) &= V\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \\ &= V\sqrt{\frac{\rho\omega}{\mu\sqrt{(1+\alpha_1^2\omega^2/\mu^2)}}}. \end{aligned} \tag{7.2-7}$$

Consequently a value of α_1 other than 0, whichever its sign, makes the capacity of the fluid body to be sheared and spun less than it is for the corresponding body of Navier-Stokes fluid but lets propagate more easily to great distances whatever shearing and spin there are. The quantity of the effect is governed by the time lapse $|\alpha_1|/\mu$, which is a constitutive property of a fluid of grade 2. A Navier-Stokes fluid has no characteristic time lapse.

EXERCISE 7.2.3 (Truesdell)
Show that if $\alpha_1 > 0$, then monotonically

$$\mathrm{a}^2 \to \frac{\rho}{\alpha_1} \text{as } \xi \to \infty, \tag{7.2-8}$$

while if $\alpha_1 < 0$, then a^2 reaches a maximum when $\varepsilon = -1/\sqrt{3}$,

$$\mathrm{a}^2_{\text{max}} = -\frac{\rho}{8\alpha_1}, \tag{7.2-9}$$

after which it falls off to zero. The frequency ω_{crit} corresponding with $\xi = -1\sqrt{3}$ is related to $a_{\max}^2$ as follows:

$$\frac{a_{\max}^2}{\omega_{\text{crit}}} = \frac{\sqrt{3}}{8}\frac{\rho}{\mu}. \tag{7.2-10}$$

Thus, in a body of fluid of grade 2 for which $\alpha_1 > 0$, the oscillations of high frequency are attenuated less than in the corresponding Navier-Stokes fluid but not so little as in any fluid of grade 2 for which $\alpha_1 < 0$. In contrast, a negative value of α_1 gives rise to a critical frequency, namely $\mu/(-\sqrt{3}\alpha_1)$, which has no counterpart in the Navier-Stokes theory. Oscillations of very high frequency are damped scarcely at all.

In contrast with the Navier-Stokes fluid, the fluid of grade 2 may produce negative stress power, as is illustrated in the following exercise.

EXERCISE 7.2.4 (Rajagopal)
Show that for the fluid of grade 2 the stress power is given by

$$w = \frac{\mu}{2}|\mathbf{A}_1|^2 + \frac{\alpha_1}{4}\,\dot{\overline{|\mathbf{A}_1|^2}} + \frac{\alpha_1 + \alpha_2}{2}\operatorname{tr}\mathbf{A}_1^3. \tag{7.2-11}$$

In the flow (2), therefore,

$$\begin{aligned} w &= \mu p^2 + \alpha_1 pq, \\ p &:= Ve^{-ax}[b\sin(\omega t - bx) - a\cos(\omega t - bx)], \\ q &:= Vwe^{-ax}[b\cos(\omega t - bx) + a\sin(\omega t - bx)]. \end{aligned} \tag{7.2-12}$$

If ρ and ω/μ are fixed positive values, and if α_1 is given a fixed value either positive or negative, there are values of x and t such that $w < 0$.

It would be wrong to discard the fluid of grade 2 because of the behavior just demonstrated, for there is no general principle or law of physics that requires the stress power to be positive always and everywhere in all motions of all fluids.

EXERCISE 7.2.5
Verify that the velocity field

$$v = a\sin\frac{n\pi x}{d}e^{\alpha t}, \tag{7.2-13}$$

with a constant and n an integer, is a solution of (1) satisfying the boundary conditions

$$v(0, t) = v(d, t) = 0 \quad \text{for all } t, \tag{7.2-14}$$

provided that α is a root of the equation

$$\frac{n^2\pi^2}{d^2} = \frac{\rho\alpha}{\mu - \alpha_1\alpha}. \tag{7.2-15}$$

The solution (13) represents an oscillation of a fluid confined by the two parallel, stationary walls $x = 0$ and $x = d$. If $\alpha_1 > 0$, there is precisely one root of (15), and it falls in the interval $(0, \mu/\alpha_1)$. Then (13) is bounded as $t \to \infty$. If $\alpha_1 < 0$, there is again one root α_0 of (15) unless $d = n\pi\sqrt{-\alpha_1/\rho}$, in which case there is none. If $d > n\pi\sqrt{-\alpha_1/\rho}$, then $\alpha_0 > 0$; if $d < n\pi\sqrt{-\alpha_1/\rho}$, then $\alpha_0 < \mu/\alpha_1$. When $d < \pi\sqrt{-\alpha_1/\rho}$, (13) is unbounded as $t \to \infty$, regardless of the value of n. If $\alpha_1 = 0$, the root of (15) is again unique; it is positive and proportional to n^2. Thus, for n sufficiently large, the rate of decay of (13) when $\alpha_1 > 0$ is slower than for the corresponding Navier-Stokes solution.

Now we seek solutions of (1) in separated variables:

$$\mathrm{v} = V(x)T(t). \tag{7.2-16}$$

Substituting this assumption into (1) yields

$$TV'' + \frac{\alpha_1}{\mu}T'V'' - \frac{\rho}{\mu}VT' = 0, \tag{7.2-17}$$

in which again we put $c = 0$. Effecting the separation delivers

$$\frac{T'}{T} = \frac{V''}{\frac{1}{\mu}(\rho V - \alpha_1 V'')} = K = \text{const.}, \tag{7.2-18}$$

so

$$\begin{gathered} T' - KT = 0, \\ \left(1 + \frac{K\alpha_1}{\mu}\right)V'' - \frac{K\rho V}{\mu} = 0. \end{gathered} \tag{7.2-19}$$

From the first member it follows that

$$T = Ae^{Kt}, \tag{7.2-20}$$

and if solutions are to be bounded as $t \to \infty$, we require that

$$K = -a^2, a > 0, \tag{7.2-21}$$

where a is a constant. Equations (20) and (21) imply that

$$V'' + \frac{\rho a^2}{\mu - a^2\alpha_1}V = 0. \tag{7.2-22}$$

If $\alpha_1 < 0$, then

$$V = B_1 \cos \lambda x + B_2 \sin \lambda x,$$
$$\lambda := \left(\frac{\rho a^2}{\mu - a^2 \alpha_1} \right)^{\frac{1}{2}}, \tag{7.2-23}$$

where B_1 and B_2 are constants. If $0 < \alpha_1 < \mu/a^2$, then

$$V = B_3 e^{\sqrt{\gamma} x} + B_4 e^{\sqrt{\gamma} x},$$
$$\gamma := \left(\frac{\rho a^2}{a^2 \alpha - \mu} \right). \tag{7.2-24}$$

Solutions of the form (16) can be generalized to provide such combinations as

$$V = Re \sum_n A_n e^{\lambda_n x} e^{\beta t} \tag{7.2-25}$$

when the sum is finite and A_n, λ_n, and β are arbitrary complex constants.

A more general solution is valid if $t > t_o$:

$$\mathrm{v} = A \int_{-\infty}^{+\infty} \exp \left[\frac{-\mu \beta^2}{\rho + \alpha_1 \beta^2} (t - t_o) \right] \left\{ \begin{matrix} \cos \beta x \\ \sin \beta x \end{matrix} \right\} d\beta, \tag{7.2-26}$$

in which A is a constant and either cos βx or sin βx appears in the integrand. Further formal solutions can be generated by using techniques from the theory of conduction of heat.[2]

So far, we have not considered any initial-value problem for equation (1). Such problems arise when we wish to describe the way a prescribed flow decays due to internal damping as time goes on.

THEOREM 7.1 (Coleman, Duffin, and Mizel[3])

If $\alpha_1 < 0$, every solution of (1) satisfying (14) is unbounded of exponential order as $t \to \infty$, except for a finite-dimensional family. If $d < \pi \sqrt{-\alpha_1/\rho}$, then the only bounded solution of (1) satisfying (14) is $\mathrm{v} \equiv 0$.

THEOREM (Coleman, Duffin, and Mizel). Let a function $\bar{\mathrm{v}}$ such that

$$\bar{\mathrm{v}}(0) = \bar{\mathrm{v}}(d) = 0 \tag{7.2-27}$$

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[2]H.S. Carslaw and J.C. Jaeger, *Conduction of Heat in Solids*, (London: Oxford University Press, 1959), and E.T. Copson, *Partial Differential Equations* London: Cambridge University Press, 1975.

[3]B.D. Coleman, R.J. Duffin, and V.J. Mizel, "Instability, uniqueness, and non-existence theorems for the equation $u_t = u_{xx} - u_{xtx}$ on a strip," *Archive for Rational Mechanics and Analysis* 19 (1965), 100–116.

be given. If $\alpha_1 < 0$ and $d = n\pi\sqrt{-\alpha_1/\rho}$ for some integer n, the condition

$$\int_0^d \bar{v}(x) \sin\left(x\sqrt{-\frac{\rho}{\alpha_1}}\right) dx = 0, \tag{7.2-28}$$

is necessary for the existence of a solution of (1) satisfying (14) and the initial condition

$$v(x,0) = \bar{v}(x) \qquad \text{for all } x \in [0,d]. \tag{7.2-29}$$

Thus, if $\alpha_1 < 0$ the class of flows (5.5-1), again when $c = 0$, includes no solutions that vanish upon $x = 0$ and $x = n\pi\sqrt{-\alpha_1/\rho}$ and satisfy the initial condition (29). Indeed, a solution of the equation of motion that satisfies (5.5-1) when $t = 0$ generally fails to satisfy it at later instants.[4]

Also in other lineal flows the behavior of the fluid of grade 2 depends strongly on the sign of α_1. Considering only fluids for which $\alpha_1 > 0$, Ting[5] in an elegant and exhaustive memoir examined several classes of solutions of (1) and of the corresponding equation for flows within circular pipes. In all circumstances he considered he proved that the problems specified by initial values and boundary values were well set, and he constructed their solutions explicitly. They differ greatly from their counterparts for the Navier-Stokes fluid both mathematically and physically. All solutions satisfy the partial differential equations up to the boundaries and upon them, with all partial derivatives continuous even at the very instant when motion or decay begins, so the "Lamb paradox" is avoided. All rates of decay and dissipation are greater than for the Navier-Stokes fluid with the same viscosity.

It is a very different matter for fluids such that $\alpha_1 < 0$; Ting remarked that for them, none of the problems he considered had a bounded general solution. That the solutions of (1) for $\alpha_1 < 0$ are rather peculiar is illuminated by an example of instability due to Coleman, Duffin, and Mizel. Let v_1 be any unbounded solution of (1) satisfying (14) as in Exercise 2.5., and let $v(x,t)$ be a bounded solution of (1) in the channel $0 < x < d$ for all time. Set

$$v^* := v + \varepsilon v_1. \tag{7.2-30}$$

Then v^* also is a solution of (1), and

$$|v^*(x,0) - v(x,0)| < \varepsilon. \tag{7.2-31}$$

Moreover, $v^*(x,t)$ is unbounded as $t \to \infty$. Consequently, if for the fluid between the two stationary plates $x = 0$ and $x = d$ some particular initial speed field $v(x,0)$ gives rise to unbounded solution $v(x,t)$, there are infinitely many initial

[4]B.D. Coleman and V.J. Mizel, "Breakdown of laminar shearing flows for second-order fluids in channels of critical width," *Zeitschrift für Mathematik und Physik* 46 (1966): 445–448.

[5]T.W. Ting, "Certain nonsteady flows of second-order fluids," *Archive for Rational Mechanics and Analysis* 14 (1963): 1–26.

speed fields $\mathrm{v}^*(x, 0)$ that differ by an arbitrarily small amount from $\mathrm{v}(x, 0)$ and give rise to unbounded solutions. In this sense the kind of internal damping represented by the fluid of grade 2 when $\alpha_1 < 0$ may amplify rather than attenuate disturbances.

The class of bounded solutions of (1) satisfying *inhomogeneous* boundary conditions is made clear by the following example:

$$\mathrm{v}(x, t) = A\left(1 - \frac{x}{d}\right) \text{ for all } t, \tag{7.2-32}$$

which satisfies

$$\mathrm{v}(0, t) = A,\ v(d, t) = 0 \text{ for all } t. \tag{7.2-33}$$

EXERCISE 7.2.6 (Rajagopal)

When c=0, show that the velocity field is given by

$$\mathrm{v}(x, t) = A\left(1 - \tfrac{x}{d}\right) - \tfrac{2A}{\pi}\sum_{n=1}^{\infty} \tfrac{1}{n} - \exp\left(\tfrac{-n^2\pi^2\beta_n t}{d^2}\right)\sin\tfrac{n\pi x}{d},$$

$$\beta_n = \frac{\mu}{\rho + \frac{\alpha_1 n^2\pi^2}{d^2}}, \tag{7.2-34}$$

is a solution of (1) if $\alpha_1 \geq 0$.

This behavior makes it plain that the fluid of grade 2 when $\alpha_1 < 0$ does not afford a good model for general flows of fluids in nature. Nevertheless, we gave reasons for regarding the fluid of grade 2 as an improvement upon the Navier-Stokes fluid, in the sense of successive approximation, even if $\alpha_1 < 0$. How can this be?

In fact, there is no contradiction. The Navier-Stokes theory emerges at the first stage of approximation for the retarded flows. Moreover, it has been found for over a century to provide solutions, usually called approximate, that *conform* excellently *with* many flows of many real fluids, though by no means all that occur in nature. In retarded flows, ultimately, the fluid of grade 2 may provide a better approximation to a general fluid than does the Navier-Stokes fluid. This fact does not give any grounds for expecting it to be a better theory for general flows.

First, it is possible that, for some particular physical fluid, the Navier-Stokes theory is really the correct one. Then $\alpha_1 = \alpha_2 = 0$, and the objection against the fluid of second-grade breaks down. This kind of explanation, however, rests on conjectured outcomes of future experiments and thus is to be regarded with suspicion. Second, the formal approximation does not provide models for general flows and gives no status to the differential equations of motion resulting from it. If $a(x) - b(x)$ is small, that does not make $a'(x) - b'(x)$ small (if it did, we could conclude that when a son reached his father's height he would necessarily stop growing rapidly). Conversely, a small term in a differential equation does not necessarily have a small effect on the solution.[6]

Third, a process of approximation by series expansion often yields successive terms that afford better approximation in ever narrower ranges at the expense of worse approximation overall.

For example, the graph of an affine function whose tangent at some point coincides with the tangent of the graph of a differentiable, monotone function at the same point shares with that function its main property, namely, being monotone and differentiable, although at distant points the error in using it as an approximation may be great. The second approximation to that same function by Taylor's theorem is a parabola, which near the point where it is made is certainly better than the tangent but far away is far worse, being in fact in error by an infinite amount at either $+\infty$ or $-\infty$ or both. Likewise, the function $y = \sin x$ is fairly well approximated by any of the lines $y = k$ provided that $|k| < 1$; although only two of these are even tangent anywhere to the curve, they all preserve two of its main properties, namely, being periodic with the same period (among others) and bounded by the same bounds. No polynomial approximation, and hence no finite number of terms in any Taylor approximation, has either of these properties. These examples show that in unbounded regions, a crude first approximation may give a better overall picture than does any higher polynomial approximation, however refined, near a single point.

The behavior of fluids of grade 2 undergoing unsteady helical flow has been determined[7] on the assumption that $\alpha_1 < 0$. A variety of unidirectional flows[8] have also been studied.

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[6]The following remarks of Truesdell (Annual Review of Fluid Mechanics (eds. M. Van Dyke and W.G. Vincenti), vol. 6, pp 111–146, Annual Reviews, California, (1974)) are particularly relevant.

> In the experimental literature the result of Coleman and Noll is sometimes given a vastly exaggerated statement such as: "For sufficiently slow flows the second-order fluid is a valid approximation to any simple fluid." Coleman and Noll neither stated nor proved any such thing. In particular, they never claimed that all simple fluids have fading memory in the sense of Coleman and Noll for it is easy to provide examples of fluids that do not. Second, while their general theory concerns many different kinds of approximation, their only application of it to "slow" flows refers to those obtained by retardation of a given flow. Other definitions of "slow" flow lead to somewhat different results. Finally, they neither claimed nor proved any relation at all between the *solutions* of the differential equations of motion for the fluid of grade 2 and the *solutions* of the equations of motion of the general simple fluid that the particular fluid of grade 2 approximates. An example known from the kinetic theory of gases shows it is risky to jump to conclusions in matters of this kind.

The approximation procedure of Coleman and Noll does not provide general models for fluids; they provide approximate expression for the stress in a simple fluid in retarded motions.

[7]H. Markovitz and B.D. Coleman, "Nonsteady helical flows of second-order fluids," *The Physics of Fluids* 7 (1964): 833–841.

[8]K.R. Rajagopal, "A Note on Unsteady unidirectional flows of a non-Newtonian Fluid," *International Journal of Non-Linear Mechanics* 17, 369–373, (1982).

7.3 Unsteady Plane Flows

A flow is *plane* if its velocity field is parallel to a certain plane and constant along each perpendicular to that plane; the plane used to define it is called the *plane of flow*. A plane isochoric flow $\mathbf{u}$, steady or not, has a stream function q:

$$\mathbf{u} = \nabla q^{\perp}. \tag{6.3-2}$$

Any function of q alone is constant along each streamline at every fixed time, so

$$f(q) = f'(q). \tag{7.3-1}$$

It follows that

$$\mathrm{w} = -\Delta q, \tag{7.3-2}$$

where Δ is the harmonic operator in the plane of flow and ω is the magnitude of the vorticity. The vorticity equation (3.8-5) collapses to a single scalar equation because the only components of its left-hand side that are not null are $\pm\rho\dot{w}$. For an incompressible fluid of grade 2, that equation is

$$\rho\dot{\mathrm{w}} = \mu\Delta\mathrm{w} + \alpha_1(\Delta\mathrm{w})^{\cdot}, \tag{7.3-3}$$

from which a partial differential equation for q follows by use of (2).

Cauchy's first law when specialized to plane flow of homogeneous incompressible fluid may be written down by inspection from the statement about steady flow at the beginning of Section 6.3. We need only put $a = 0$ there, note that φ becomes a function of p and t, and restore the local acceleration in (6.3-5), so obtaining from (6.3-12)

$$\rho(\mathbf{u}' + (\nabla\mathbf{u})\mathbf{u}) = -\nabla\varphi + \operatorname{div}\mathbf{\Pi}, \tag{7.3-4}$$

in which $\mathbf{\Pi}$, the plane tensor of determinate stress, is to be expressed by means of the constitutive relation of the fluid. The meaning of (3) as a condition of integrability assures us that given a solution w, we may integrate the appropriate special instance of (4) and so determine the corresponding φ. In *preceding sections* we have seen examples of such a procedure for other classes of flows.

For some purposes it is enough to know the tractions on surfaces to which the fluid adheres. For a fluid of grade 2 those are determined explicitly by w, w$'$, and the constitutive constants through (6.1-25).

While (3) is generally a nonlinear partial differential equation, there are kinematical assumptions under which it reduces to a linear one. Namely, if both w and Δw are constant along each streamline at every fixed time, we may apply (1) and so obtain

$$\rho\mathrm{w}' = \mu\Delta\mathrm{w} + \alpha_1\mathrm{w}'. \tag{7.3-5}$$

Equivalently, a function q such that

$$\Delta q = f(q), \ \Delta\Delta q = g(q), \ \rho\Delta q' = \mu\Delta\Delta q + \alpha_1\Delta\Delta q' \tag{7.3-6}$$

is the stream function of a plane flow of an incompressible fluid of grade 2. Solutions of this kind are unusual in that while merely a linear equation needs to be solved in order to obtain them, they may correspond with nonlinear dynamics: their convective accelerations need not vanish. Thus they offer examples of the combined effects of inertia and viscosity that are precise yet not trivial; nevertheless, they are not typical of such interactions in more general flows.

EXERCISE 7.3.1

Show that a solution of $(6)_{1,2}$ is given by

$$q = Re\sum_n C_n e^{\Gamma_n x_1} e^{\beta_n x_2} e^{\lambda t}, \tag{7.3-7}$$

in which x_1 and x_2 are Cartesian coordinates, the sum is finite, and C_n, Γ_n, β_n, and λ are arbitrary constants.

The following exercises take up several solutions of $(6)_3$ which have the form (7).

EXERCISE 7.3.2 (Rajagopal and Gupta)

For arbitrary A, m, and n, show that

$$q = A\cos mx_1 \cos nx_2 e^{-\lambda t} \tag{7.3-8}$$

satisfies $(6)_3$ if

$$\frac{1}{\lambda} = \frac{\alpha_1}{\mu} + \frac{c}{\mu(m^2+n^2)} \tag{7.3-9}$$

This flow presents a regular array of vortices within contiguous rectangular cells whose boundaries remain at rest. The directions of the flows are opposite on the two sides; the fluid is at rest not only on the boundaries of the cells but also at their centers. The lengths of the cells in the x_1-direction are π/m; in the x_2-direction π/n; while λ determines whether the vortices decay or grow. When $\alpha_1 = 0$, this solution reduces to one found by G.I. Taylor in the Navier-Stokes theory. Then $\lambda > 0$, so the vortices decay. For the fluid of grade 2 they decay if $\alpha_1 > -\rho/(m^2+n^2)$; they grow if the inequality is reversed; if equality holds, the solution reduces to a state of rest.

EXERCISE 7.3.3 (Rajagopal and Gupta)
For arbitrary constants A, a, and b, verify that

$$q = A \sinh ax_1 \sinh bx_2 e^{\lambda t} \tag{7.3-10}$$

satisfies $(6)_3$ if

$$\lambda = \frac{\mu(a^2+b^2)}{\rho - \alpha_1(a^2+b^2)}. \tag{7.3-11}$$

For interpretation of (10) we take the constants A, a, and b as positive. The flow then corresponds to two oppositely directed streams coming from $x_1 = \pm\infty$ to impinge upon the plane $x_1 = 0$, which turns them at right angles and directs them toward $x_2 = \pm\infty$. The dividing streamlines are the straight lines $x_1 = 0$ and $x_2 = 0$; on the latter, the fluid travels straight toward the origin; on the former, straight away from the origin, in opposite directions. When $\alpha_1 = 0$, this flow reduces to a flow of the Navier-Stokes fluid, as noticed by C.Y. Wang; then $\lambda > 0$, and hence the speed on all streamlines tends monotonically to infinity. If $\alpha_1 < \rho/(a^2+b^2)$, and in particular if $\alpha_1 < 0$, again $\lambda > 0$, so the speed again increases with time. If, on the contrary, $\alpha_1 > \rho/(a^2+b^2)$, then $\lambda < 0$, and the flow tends with time to a state of rest.

EXERCISE 7.3.4 (Rajagopal and Gupta)
Show that if

$$q = A \cosh ax_1 \cos\, bx_2 e^{\lambda t} \tag{7.3-12}$$

is to satisfy $(6)_3$, any of the following three conditions is sufficient:

(i) $a^2 = b^2$ with arbitrary λ, μ, α_1;

(ii) $\mu = 0$, $\rho = \alpha_1(a^2 - b^2) > 0$, λ arbitrary;

(iii) $\mu \neq 0$, $a^2 \neq b^2$,

$$\lambda = \frac{\mu(a^2-b^2)}{\rho - \alpha_1(a^2-b^2)}, \tag{7.3-13}$$

and that collectively they provide a necessary condition for (12) to satisfy $(6)_3$.

The first of the alternatives derived in this exercise is the only one possible in an inviscid fluid. In that context it was noticed and studied by Kelvin[9] that they

[9]William Thompson, later Lord Kelvin (Baron and Kelvin of Lanrgs) *Mathematical and Physical Papers*, vol. 4 (Cambridge: Cambridge University Press, 1910), 186.

were divided by "a cat's-eye border pattern of elliptic whirls." For inviscid and linearly viscous fluids, (12) remains a solution even if $\lambda \neq 0$, but the fact that it may increase or decrease exponentially in time makes it seem odd and suggests that it might be unstable. The third alternative, in which λ is determined by parameters characteristic of the fluid and of the flow, was noticed by C.Y. Wang for a Navier-Stokes fluid. The lines $x_2 = \frac{1}{2}\pi/b$ are steady streamlines, the speeds of flow along which tend to infinity as $x_1 \to \pm\infty$ at any fixed t. For a Navier-Stokes fluid, the speeds grow to infinity with time if $a^2 > b^2$, decay to 0 if $a^2 < b^2$. Those solutions may provide a class of perturbations of a parallel flow. For a fluid of grade 2 the behavior of the solution in time depends also on the sign and magnitude of α_1, as is plain from (13). The alternatives of growth and decay may be contrary to those that the Navier-Stokes theory requires.

7.4 Steady Plane Flows

Most of the particular precise flows adduced as examples in books on the mechanics of fluids are monotonous; many of them are viscometric. Next common are steady instances of the two kinds we studied in the preceding sections. For them the stream function satisfies the following differential equation:

$$\rho \nabla \Delta q \cdot \nabla q^{\perp} = \mu \Delta\Delta q + \alpha_1 \nabla \Delta\Delta q \cdot \nabla q^{\perp}. \tag{7.4-1}$$

EXERCISE 7.4.1

Verify that substituting (63-2) into (5.4-1) yields (1). For the Navier-Stokes theory, *i.e.*, when $\alpha_1 = 0$ and $\mu > 0$, (1) has been studied in great detail.

Jeffery and Hamel sought rotational flows having the same streamlines as does some irrotational flow. Hamel proved that such streamlines had to be logarithmic spirals, which may degenerate into concentric circles. A theorem of this kind is far from being a mere rediscovery of a trivial solution. Its objective is different: to illuminate the meaning of an observed flow pattern. The theorem of Jeffery and Hamel shows that if the streamlines determined by an experiment are not logarithmic spirals yet do correspond with some isochoric irrotational flow, then the observed flow if interpreted as belonging to a solution of the Navier-Stokes theory must be not only universal but also exempt from the effects of viscosity.

Kaloni and Huschilt have analyzed the corresponding problem when $\alpha_1 \neq 0$. Following them, we seek solution of (1) such that

$$q = F(\alpha), \quad \Delta\alpha = 0, \quad \Delta q \neq 0. \tag{7.4-2}$$

Denoting differentiation with respect to the argument of F by a prime, we conclude that

$$F''(\alpha) \neq 0. \tag{7.4-3}$$

Let β denote the conjugate harmonic of α; the trajectories of α and β then provide an orthogonal coordinate system. Substituting (2) into (1) yields

$$(\mu + 2\alpha_1 JbF)\left[F^{iv} + 2aF''' + (a^2+b^2)F''\right] - \rho F'F'' = 0, \tag{7.4-4}$$

where J is the Jacobian of α and β with respect to x_1 and x_2. Kaloni and Huschilt show that the streamlines determined by the solution of (4) when $\alpha_1 \neq 0$ coincide with the curves

$$\log r = \text{constant}, \quad r^2 := x_1^2 + x_2^2 \tag{7.4-5}$$

and hence are concentric circles.

Next, again following Kaloni and Huschilt, we consider solutions of (1) of the form

$$q = x_2 F(x_1) + G(x_1). \tag{7.4-6}$$

Substituting (6) into (1), we obtain the following differential equations for F and G:

$$\mu F^{\text{iv}} - \alpha_1\left[F'F^{\text{iv}} - FF^{\text{v}}\right] + \rho\left[F'F'' - FF'''\right] = 0, \tag{7.4-7}$$

$$\mu G^{\text{iv}} - \alpha_1\left[G'F^{\text{iv}} - FG^{\text{v}}\right] + \rho\left[G'F'' - FG'''\right] = 0. \tag{7.4-8}$$

These equations can be integrated once to give

$$\mu F''' - \alpha_1\left[-FF^{\text{iv}} + 2F'F''' - F''^2\right] + \rho\left[F'^2 - FF''\right] = A, \tag{7.4-9}$$

$$\mu G''' - \alpha_1\left[-GG^{iv} + F'G''' - F''G'' + F'''G'\right] + \rho\left[F'G' - FG''\right] = B, \tag{7.4-10}$$

where A and B are constants. Kaloni and Huschilt show that if the body force field is lamellar and the pressure remains bounded, necessarily $A = B = 0$. Then

$$\mu F''' - \alpha_1\left[-FF^{\text{iv}} + 2F'F'' - F''^2\right] + \rho\left[F'^2 - FF''\right] = 0, \tag{7.4-11}$$

$$\mu G''' - \alpha_1\left[-GG^{\text{iv}} + F'G''' - F''G'' + F'''G'\right] + \rho\left[F'G' - FG''\right] = 0. \tag{7.4-12}$$

These equations have been studied at length by Berker for the Navier-Stokes theory, for which $\alpha_1 = 0$. For the fluid of grade 2, the presence of fourth derivatives suggests that some condition beyond those that Berker used may be prescribed.

The nonlinear equation (7) admits the solution

$$F = \frac{\mu a}{(\rho - \alpha_1 a^2)}(1 + ke^{ax_1}), \tag{7.4-13}$$

in which k and a are constants.

EXERCISE 7.4.2 (Kaloni and Huschilt)
When $G := 0$, show that the solution is

$$q = x_2 \left[\frac{\mu a}{(\rho - \alpha_1 a^2)}(1 + ke^{ax_1}) \right]. \tag{7.4-14}$$

If $k = -1$,

$$\dot{x}_1 = \frac{-\mu a}{(\rho - \alpha_1 a^2)}(e^{ax_1} - 1), \tag{7.4-15}$$

and

$$\dot{x}_2 = \frac{\mu a^2}{(\rho - \alpha_1 a^2)} e^{ax_1} x_2. \tag{7.4-16}$$

If a < 0 and $\rho > \alpha_1 a^2$, streams of fluid come toward the plane $x_2 = 0$ from opposite directions; if, on the contrary, $\rho < a_1 a^2$, the direction of the flow is reversed. When $a = 0$, only the former flow, which corresponds to the two streams impinging on the $x_2 = 0$ plane, is possible.

We now consider some solutions of (6) when $G := 0$. Substituting (14) into (8), we find that

$$\begin{aligned} G^{iv} &- \frac{\alpha_1 a}{\rho - \alpha_1 a^2}\left[G' a^4 k e^{ax_1} + (1 + ke^{ax_1})G^{v}\right] \\ &+ \frac{\rho a}{\rho - \alpha_1 a^2}\left[G' \mathrm{a}^2 k e^{ax_1} - (1 + ke^{ax_1})G'''\right] = 0. \end{aligned} \tag{7.4-17}$$

Kaloni and Huschilt show that when $a = 1$, this equation can be reduced after one integration to a hypergeometric equation by means of the substitution

$$R(x_1) = e^{-x_1}(G'' - G'), \tag{7.4-18}$$

provided the constant of integration is taken as zero. In this case it is easy to write down the appropriate expression for q. It does not seem easy to find the general solution of (17). Nevertheless, the inferences we have drawn about solutions of the form (6) illustrate what can be learned by investigating hypotheses that reduce the equations of motion for a Navier-Stokes fluid and its counterpart fluids of grade 2 to a system of ordinary differential equations, as well as what remains to be learned about this particular class of solutions.

To conclude this section, we analyze a simple flow that illustrates the difficulty of specifying boundary conditions for fluids of grade 2. The problem also provides another example of the intricate effects of the sign of α_1 upon the existence of solutions in the theory of fluids of grade 2.

Following Rajagopal and Gupta, we consider the shearing of a fluid of grade 2 by an infinite porous plate that provides suction or injection. To this end we study the following class of isochoric velocity fields:

$$\dot{x}_1 = u(x_2), \quad \dot{x}_2 = v = \text{const.} \neq 0, \quad \dot{x}_3 = 0, \tag{7.4-19}$$

and we take the plane $x_2 = 0$ as representing the plate. The condition of adherence to that plate is

$$u(0) = 0, \tag{7.4-20}$$

and

$$\left.\begin{matrix} v > 0 \\ v < 0 \end{matrix}\right\} \text{ represents } \left\{\begin{matrix} \text{injection.} \\ \text{suction.} \end{matrix}\right. \tag{7.4-21}$$

Plainly

$$q = -\int^{x_2} u(x)\,dx + vx_1, \tag{7.4-22}$$

so from (1) we obtain an equation for u which when integrated twice yields

$$\mu u' + \alpha_1 v u'' - \rho v u = -K\rho v x + \mu K - \rho v L, \tag{7.4-23}$$

where K and L are arbitrary constants. Here and henceforth in this section, a prime denotes differentiation with respect to x_2, which we abbreviate to x.

EXERCISE 7.4.3
Proceeding as at the beginning of Section 5.4 show that

$$\rho\varphi = (2\alpha_1 + \alpha_2)\mathrm{u}'^2 - K\rho v \mathrm{x}_1 + \text{const.} \tag{7.4-24}$$

For the Navier-Stokes theory, which obtains when $\alpha_1 = 0$, the solution of (23) is given as follows in terms of three arbitrary constants C, K, L:

$$u = Ce^{\rho v x/\mu} + Kx + L; \tag{7.4-25}$$

if $\alpha_1 \neq 0$, in contrast, a fourth arbitrary constant appears:

$$u = C_1 \mathrm{e}^{\mathrm{m}_1 x} + C_2 \mathrm{e}^{\mathrm{m}_2 x} + Kx + L,$$

$$2\mathrm{m}_1, 2\mathrm{m}_2 = -\frac{\mu}{\alpha_1 v} \pm \sqrt{\left(\frac{\mu}{\alpha_1 v}\right)^2 + \frac{4\rho}{\alpha_1}}, \tag{7.4-26}$$

except that if $m_1 = m_2 = m$, say, then necessarily $\alpha_1 < 0$, $m = 2\rho v/\mu$, and

$$u = C_1 e^{mx} + C_2 x e^{mx} + Kx + L. \tag{7.4-27}$$

Here the notation is appropriate to values of μ, α_1, and v such as to make the square root real, but we shall easily read off a corresponding statement when m_1 and m_2 are complex conjugates.

The boundary condition (20) reduces (25), $(26)_1$, and (27) to

$$\begin{aligned} u &= C\left(1 - e^{\rho v x/\mu}\right) + Kx, \\ u &= C_1\left(1 - e^{m_1 x}\right) + C_2\left(1 - e^{m_2 x}\right) + Kx, \\ u &= C_1\left(1 - e^{mx}\right) + C_2 x e^{mx} + Kx, \end{aligned} \tag{7.4-28}$$

respectively.

EXERCISE 7.4.4

Considering the signs of the terms in $(26)_2$ and (27) show that the following alternatives are exhaustive if $\alpha_1 \neq 0$.

1. If $\alpha_1 > 0$, then $m_1 > 0, m_2 < 0$.
2. If $\alpha_1 < 0$ and $v > 0$, then $\operatorname{Re} m_1 > 0$, $\operatorname{Re}\ m_2 > 0$.
3. If $\alpha_1 < 0$ and $v < 0$, and if $-\alpha_1 \neq \mu^2/(4\rho v^2)$, then $\operatorname{Re} m_1 < 0$, $\operatorname{Re} m_2 < 0$.
4. If $-\alpha_1 = \mu^2/(4\rho v^2)$, then $\operatorname{sgn} m = \operatorname{sgn} v$.

We wish to consider only solutions that correspond with a uniform stream at infinity. That is,

$$u(x) \to u_\infty, \text{ say, as } x \to \infty. \tag{7.4-29}$$

For this condition to hold, it is necessary that the coefficient of each exponential in (25), (26), and (27) be 0 if the real part of the exponent is positive; if that is so, then (29) requires that $K = 0$, and from (24) we see that $\partial_{x_1}\phi = 0$, so

$$p + \rho\varpi = (2\alpha_1 + \alpha_2)u'^2 + \text{const.} \tag{7.4-30}$$

First we consider the Navier-Stokes theory, to which $(28)_1$ corresponds. If $v > 0$, no choice of the constants in (28) can satisfy (29), so *there is no solution corresponding with injection*. If $v < 0$,

$$u = u_\infty\left(1 - e^{\rho v x/\mu}\right). \tag{7.4-31}$$

The vorticity $w = -u' = -u_\infty(\rho v/\mu)\exp(\rho v x/\mu)$; hence w is greatest upon the plate $x = 0$, namely $-u_\infty(\rho v/\mu)$, and it decays rapidly with the distance from the plate. The flow is not only irrotational at ∞ but also virtually irrotational in the part of the fluid beyond a *boundary layer*, the thickness of which may be taken conveniently as the logarithmic decrement $\mu/(-\rho v)$. The statements preceding (31) are used in boundary-layer theory to represent the flow downstream from the leading edge of a semiinfinite flat plate. There they are taken to mean that suction subdues the growth of the boundary layer while injection so destabilizes the flow that no solution exists.

The fluid of grade 2 allows broader possibilities; in particular, it provides well-behaved solutions corresponding to injection. We can read off the solutions by applying to $(28)_{2,3}$ the statements in Exercise 4.4, at the same time imposing the limiting requirement (29).

If $\alpha_1 > 0$, we conclude from the first alternative in Exercise 4.4 that $\mathrm{m}_2 < 0$ hence both if $v > 0$ and if $v < 0$:

$$u = u_\infty \left(1 - \mathrm{e}^{\mathrm{m}_2 x}\right). \tag{7.4-32}$$

Thus, if $\alpha_1 > 0$, the fluid of grade 2 provides solutions, both for suction and for injection, and they are of the boundary-layer kind; the thickness may be taken as $-1/\mathrm{m}_2$. If $\alpha_1 < 0$ and $v > 0$, *there is no solution* of the form (19) corresponding to the conditions (2) and (29). If $\alpha_1 < 0$ and $\mathrm{v} < 0$, *there are infinitely many solutions*. These decay monotonically if $4\rho v^2 < \mu^2/(-\alpha_1)$ but are damped sinusoidal oscillations in x if $4\rho v^2 > \mu^2/(-\alpha_1)$.

Differences such as these often fail to be revealed by procedures of perturbation.

EXERCISE 7.4.5
If $4\rho v^2 = \mu^2/(-\alpha_1)$, show that there is no solution if $v < 0$, while there are infinitely many if $v > 0$.

The student will notice from (28) that if the conditions (20) and (29) are satisfied, then

$$u'(x), u''(x), u'''(x), \ldots \to 0 \quad \text{as } x \to \infty. \tag{7.4-33}$$

In other words, all solutions of the problem are very smooth at ∞; in particular, both the shearing and the shearing stress tend to 0. It follows from (30) that for these solutions

$$p + \rho\varpi \to \text{const. as x} \to \infty. \tag{7.4-34}$$

In all the foregoing treatment of the flows (19), the explicit integration that the constitutive equation of the fluid of grade 2 allows have delivered the conclusions. In particular, the ordinary differential equation (23) is linear. If a more general kind of fluid is considered, the counterpart of (23) will usually be nonlinear, yet sometimes similar conclusions may be obtained by appeal to general arguments about ordinary differential equations. A relatively simple example treated by Rajagopal, Szeri, and Troy[10] illustrates the great difficulties that impede proof of such conclusions, even when more smoothness than seems physically reasonable is assumed of the solutions of the differential equations.

EXERCISE 7.4.6 (Steady Axisymmetric Flows)
In cylindrical coordinates, the flow represented by

$$\dot{r} = v_1(r, z), \qquad \dot{\theta} = 0, \qquad \dot{z} = v_3(r, z) \tag{7.4-35}$$

axisymmetric. Show that a stream function q can be defined such that

$$\dot{r} = \frac{1}{r}\partial_z q, \quad \dot{z} = -\frac{1}{r}\partial_r q. \tag{7.4-36}$$

If

$$E^2 := \partial_r^2 - \frac{1}{r^2}\partial_r + \partial_z^2, \tag{7.4-37}$$

show that for an irrotational flow $E^2 q = 0$. Show that, in general

$$\rho\left[\partial_r q \partial_z\left(\frac{E^2 q}{r^2}\right) - \partial_z\partial_r\left(\frac{E^2 q}{r^2}\right)\right] + \frac{\mu E^4 q}{r} - \alpha_1\left[\partial_r q \partial_z\left(\frac{E^4 q}{r^4}\right) - \partial_z q \partial_r\left(\frac{E^4 q}{r^2}\right)\right]$$
$$+ \frac{2(\alpha_1+\alpha_2)}{r}\left\{\left[\partial_z q E^2\left(\frac{E^2 q}{r^2}\right) + 2\left[\partial_z^2 q \partial_z\left(\frac{E^2 q}{r^2}\right) + \partial_z\partial_r q \partial_r\left(\frac{E^2 q}{r^2}\right)\right]\right\} = 0. \tag{7.4-38}$$

7.5 Steady Flow between Rotating Parallel Plates

Following the work of Berker and Rajagopal, we now apply the apparatus set up in Section 5.5 for a body of fluid confined between parallel plates that rotate with constant and equal angular speeds ω about a common axis or about distinct parallel axes. Components of the velocity field are

$$\dot{x}_1 = -\Omega(x_2 - g(x_3)), \quad \dot{x}_2 = \Omega(x_1 - f(x_3)), \quad \dot{x}_3 = 0, \Omega = \text{const.} \tag{4.2-35}$$

The functions h_1 and h_2 defined following (5.6-3) are determined as follows from the constitutive relation (6.1-17):

$$\begin{aligned} h_1 &= \mu\Omega g'' - \alpha_1\Omega^2 f'', \\ h_2 &= -\mu\Omega f'' - \alpha_1\Omega^2 g'', \end{aligned} \tag{7.5-1}$$

[10] K.R. Rajagopal, A.Z. Szeri, and W. Troy, "An existence theorem for the flow of a non-Newtonian fluid past an infinite porous plate," *International Journal of Non-Linear Mechanics* 21 (1986): 279–89.

so the coefficient α_2 has no effect upon the velocity field. Comparison of (1) with (5.6-6) yields

$$-\mu\Omega g'' + \alpha_1\Omega^2 f'' + \rho\Omega^2 f + A = 0,$$
$$\mu\Omega f'' + \alpha_1\Omega^2 g'' + \rho\Omega^2 g + B = 0, \tag{7.5-2}$$

where A and B are assignable functions of t.

We first consider only the instance in which the axes of rotation coincide, so in (5.6-10) we may set $a = 0$:

$$f(d) = f(-d) = g(d) = g(-d) = 0. \tag{7.5-3}$$

Of course boundary conditions do not suffice to determine a unique solution of (2) because that system expresses only the vorticity equation and hence corresponds to infinitely many flows. In our starting point, the velocity field (4.2-35), the two centers of rotation are made to lie in the loci $x_2 = g(x_3)$ and $x_1 = f(x_3)$, respectively. Here we consider a single axis only, and there is no loss in generality if we assume that it intersects the plane $x_3 = 0$ at $(\ell, 0, 0)$. Then

$$f(0) = \ell, \qquad g(0) = 0. \tag{7.5-4}$$

Under the assumptions $A = 0$, $B = 0$, the following exercises provide a solution of the linear differential system of (2) that satisfies the six conditions (3) and (4).

EXERCISE 7.5.1
Show that

$$f(x_3) = \frac{\ell}{\delta}\Big[(\cosh md\ \cos nd - \cosh mx_3 \cos nx_3)(\cosh md\ \cos nd - 1)$$
$$+ (\sinh md\ \sin nd - \sinh mx_3\ \sin nx_3)(\sinh md\ \sin nd)\Big],$$
$$g(x_3) = \frac{\ell}{\delta}\Big[(\sinh md\ \sin nd - \sin nx_3 \sinh mx_3)(\cosh nd \cos nd - 1)$$
$$- (\cosh md\ \cos nd - \cosh mx_3\ \cos nx_3)(\sinh md\ \sin nd)\Big],$$
$$m^2 = \frac{\rho\{[(\mu/\Omega)^2 + \alpha_1^2]^{1/2} - \alpha_1\}}{2[(\mu/\Omega)^2 + \alpha_1^2]},\ n^2 = \frac{\rho\{[(\mu/\Omega)^2 + \alpha_1^2]^{1/2} + \alpha_1\}}{2[(\mu/\Omega)^2 + \alpha_1^2]},$$
$$\delta = (\cosh md\ \cos nd - 1)^2 + (\sin md\ \sin nd)^2. \tag{7.5-5}$$

Next we consider a particular solution for which the axes of rotation of the two plates may be distinct. This time we shall prejudice the problem by studying only solutions that are symmetric about the plane halfway between the two plates. Then f and g are both even functions.

EXERCISE 7.5.2

Under the foregoing assumptions, show that

$$f''(0) = \frac{(\mu/\rho)\frac{1}{2}\Omega a}{(\mu/\rho)^2 + (\alpha_1\Omega/\rho)^2},$$
$$g''(0) = \frac{\frac{1}{2}\alpha_1\Omega^2 a/\rho}{(\mu/\rho)^2 + (\alpha_1\Omega/\rho)^2}. \qquad (7.5\text{-}6)$$

A solution of (2) subject to the conditions (3), (4), and (6) is

$$\begin{aligned} f(x_3) &= \frac{2a}{\Delta}\{\sin\, nd \cosh md[\cos nx_3 \sinh mx_3 \\ &\quad + \cos n(x_3 - d)\sinh m(x_3 - d)] \\ &\quad - \cos nd \sinh md[\sin nx_3 \cosh mx_3 \\ &\quad + \sin(x_3 - d)\cosh m(x_3 - d)]\}, \\ g(x_3) &= \frac{2a}{\Delta}\{\cos nd \sin md[\cos nx_3 \sinh mx_3 \\ &\quad + \cos n(x_3 - d)\sinh m(x_3 - d)] \\ &\quad + \sin nd \cosh md[\sin n_3 \cosh mx_3 \\ &\quad + \sin n(x_3 - d)\cosh m(x_3 - d)]\}, \\ \Delta &= 4(\sinh^2 md + \sin^2 nd), \end{aligned} \qquad (7.5\text{-}7)$$

while m and n are given by $(5)_{3,4}$.

EXERCISE 7.5.3

$$h_3 = (2\alpha_1 + \alpha_2)\Omega^2(f'^2 + g'^2), \qquad (7.5\text{-}8)$$

and the components of the traction vector on the upper plate $x_3 = d$ are

$$\begin{aligned} t_{x_1} &= -\mu\Omega g'(d) + \alpha_1\Omega^2 f'(d), \\ t_{x_2} &= \mu\Omega f'(d) + \alpha_1\Omega^2 g'(d). \end{aligned} \qquad (7.5\text{-}9)$$

Thus

$$\begin{aligned} \mu &= \frac{1}{\kappa^2\Omega}\left[t_{x_2} f'(d) + t_{x_1} g'(d)\right], \\ \alpha_1 &= \frac{1}{\kappa^2\Omega}\left[t_{x_1} f'(d) - t_{x_2} g'(d)\right], \\ \kappa^2 &= f'(d)^2 + g'(d)^2. \end{aligned} \qquad (7.5\text{-}10)$$

The locus of the centers of rotation, $x_1 = f(x_3)$ and $x_2 = g(x_3)$ has been calculated numerically for values that seem to be of physical interest. Instead of the straight line given by the "approximate" formula (5.6-11), it has the form of an S stretched to left and right. The central part is nearly a line perpendicular to the plates; it represents a core in nearly rigid rotation. To either side of this line lie the stretched top and bottom parts of the S, which represent narrow boundary layers next to the plate.

As Ω is made to increase, the boundary layers grow thinner.

The foregoing problem can be solved easily even if $A \neq 0$ and $B \neq 0$. The only change is a slight modification of (6).

Also, the problem can be modified to allow the plates to be porous and to represent suction or injection through the plates $x_3 = \pm d$, we may suppose a uniform velocity field $\dot{x} = \text{const}$.

7.6 Stability and Instability of Flows in a Closed Vessel

In Section 6.6 we established or described some stabilities and instabilities of lineal flows of a fluid of grade 2, the former on the assumption that $\alpha_1 > 0$ and the latter on the assumption that $\alpha_1 < 0$. Further examples of stabilities and instabilities were determined in some of the preceding sections of this chapter. Here we consider stability of some flows in a three-dimensional region V, supposing that there is no power supplied to the body.[11] This allows us to conclude that

$$(2/\rho)\dot{K} + \int_V \mathbf{T} \cdot \mathbf{A}_1 dV = 0, \tag{7.6-1}$$

where K is the kinetic energy of the body in the region V. Giving $\mathbf{T}$ its determination for a homogeneous incompressible fluid of grade 2, namely (6.1-17), from (1) we easily derive the *equation of energy*:

$$2\dot{K} + 1/2\alpha_1 \left(\int_V |\mathbf{A}_1|^2 dV \right)^{\cdot} + \mu \int_V |\mathbf{A}_1|^2 dV + (\alpha_1 + \alpha_2) \int_V \operatorname{tr} \mathbf{A}_1^3 dV = 0. \tag{7.6-2}$$

Hence if

$$E := 2K + 1/2\alpha_1 \int_V |\mathbf{A}_1|^2 dV, \tag{7.6-3}$$

[11] The reader is referred to I.14 of C. Truesdell, *A First Course in Rational Continuum Mechanics* (New York, Academic Press, 1991) for a detailed discussion on power, kinetic energy, and potential energy.

then

$$\dot{E} = -\mu \int_V |\mathbf{A}_1|^2 dV - (\alpha_1 + \alpha_2) \int_V \operatorname{tr} \mathbf{A}_1^3 dV. \tag{7.6-4}$$

To obtain (4), we have assumed that the power supplied is zero. An instance is provided by the flow in the closure of a simply connected, fit region to which the fluid adheres:

$$\dot{\mathbf{x}} = \mathbf{0} \text{ on } \partial V, \tag{1.8-8}$$

$0 \leq t < \infty$. Assuming the body force to be lamellar, we can easily show that the power supplied is zero.

The special instance of (2) in which $\alpha_1 = \alpha_2 = 0$, which corresponds with the Navier-Stokes theory, was studied by T.Y. Thomas and E. Hopf in the first attempts to establish rigorously the stability of flows of Navier-Stokes fluids; Serrin made it the basis of his classic and definitive analysis, which we outline below in Section 8.13. Here we present two extensions of Serrin's work, one on the assumption that $\alpha_1 < 0$, the other that $\alpha_1 > 0$, with very different conclusions for the different signs.

EXERCISE 7.6.1 (Fosdick and Rajagopal)
Let **A** be any traceless, symmetric tensor, and let α be any real number. Then, show that

$$-\frac{|\alpha|}{\sqrt{6}}|\mathbf{A}|^3 \leq \alpha \operatorname{tr} \mathbf{A}^3 \leq \frac{|\alpha|}{\sqrt{6}}|\mathbf{A}|^3. \tag{7.6-5}$$

EXERCISE 7.6.2
Let **v** be a solenoidal, twice continuously differentiable vector field in V that vanishes on ∂V. Then, show that

$$\int_V |\nabla \mathbf{v}|^2 dV = 2 \int_V |\operatorname{sym} \nabla \mathbf{v}|^2 dV = 2 \int_V |\operatorname{skw} \nabla \mathbf{v}|^2 dV. \tag{7.6-6}$$

EXERCISE 7.6.3 (Poincaré inequality)
Let **v** be as in the preceding exercise. Then, show that there is a positive constant C_P, depending only on V, such that

$$\int_V \mathbf{v}^2 dV \leq C_P \int_V |\nabla \mathbf{v}|^2 dV. \tag{7.6-7}$$

Also $C_P \to 0$ as $V \to 0$. Henceforth we assume that (1.8-8) holds.

THEOREM (Fosdick and Rajagopal). *Let a homogeneous incompressible fluid of grade 2 for which* $\mu > 0$, $\alpha_1 < 0$, *undergo a flow in a stationary vessel as specified. Then for any given positive constant M there is a time at which*

$$\int_V |\mathbf{A}_1|^3 dV > M, \tag{7.6-8}$$

provided that both V and $-\alpha_1|\alpha_1 + \alpha_2|/\mu$ *are sufficiently small.*

This theorem complements the statements of Ting and Coleman and Mizel for lineal flows by showing that a body of this kind, when flowing in a closed, stationary container that it fills, will exhibit arbitrarily large stretching as time goes on. The quantity $-\alpha_1|\alpha_1 + \alpha_2|/\mu$ is the reciprocal of a characteristic time of the fluid.

PROOF From (4), (5), and $(6)_1$ it follows that

$$\dot{E} \leq -2\mu \int_V |\mathbf{G}|^2 dV + \frac{|\alpha_1 + \alpha_2|}{\sqrt{6}} \int_V |\mathbf{A}_1|^3 dV, \tag{7.6-9}$$

and hence for any positive number τ

$$\tau\dot{E} - E \leq \frac{\tau|\alpha_1 + \alpha_2|}{\sqrt{6}} \int_V |\mathbf{A}_1|^3 dV - (2\mu\tau + \alpha_1) \int_V |\mathbf{G}|^2 dV, \tag{7.6-10}$$

while applying (5) to (7) shows that

$$E \leq (\alpha_1 + \rho C_P) \int_V |\mathbf{G}|^2 dV. \tag{7.6-11}$$

Thus far, we have not prejudiced the sign of α_1. If $\alpha_1 < 0$, we can choose τ as the reciprocal of a positive characterisitic time of the fluid,

$$\tau = \frac{-\alpha_1}{2\mu}, \tag{7.6-12}$$

and so obtain

$$\frac{-\alpha_1}{2\mu}\dot{E} - E \leq \frac{-\alpha_1|\alpha_1 + \alpha_2|}{2\sqrt{6}\mu} \int_V |\mathbf{A}_1|^3 dV. \tag{7.6-13}$$

If the asserted conclusion (8) is false, then

$$\frac{-\alpha_1}{2\mu}\dot{E} - E \leq \frac{-\alpha_1|\alpha_1 + \alpha_2|}{2\sqrt{6}\mu} M, \tag{7.6-14}$$

integration of which yields

$$E(t) \leq \left[E(0) + \frac{-\alpha_1|\alpha_1 + \alpha_2|}{2\sqrt{6}\mu} M\right] e^{2\mu t/(-\alpha_1)}. \tag{7.6-15}$$

Now turning back to (11), we see that if $\rho C_P < -\alpha_1$, then $E(t) < 0$ for all t such that $|\mathbf{G}|^2$ does not vanish almost everywhere, and in particular $E(0) < 0$. Since C_P can be made arbitrarily small by taking V small enough, $E(0) < 0$ for a region of sufficiently small volume in which $|\mathbf{G}|^2$ does not vanish almost everywhere initially. Consequently for a sufficiently small volume of fluid such that $-\alpha_1|\alpha_1 + \alpha_2|/\mu$ is sufficiently small, the quantity in square brackets in (15) is negative, and $E(t) \to -\infty$ as $t \to \infty$. From (3), then

$$\int_V |\mathbf{A}_1|^2 dV \to \infty. \tag{7.6-16}$$

EXERCISE 7.6.4
From (16) it follows that

$$\int_V |\mathbf{A}_1|^3 dV \to \infty. \tag{7.6-17}$$

In order to obtain (17), we assumed that (8) was false. Because (17) contradicts the negation of (8), (9) itself is true.

Experimenters who correlate their measurements on fluids with mathematical conclusions from the theory of the incompressible fluid of grade 2 claim that to get agreement they must assign negative values to α_1. On the other hand, experimenters usually regard an unstable flow as something that does not occur in nature. In view of the instabilities displayed here and in preceding sections of this chapter, it would seem that the fluids tested by experimenters cannot be modeled by the fluid of grade 2 at all, and therefore data adduced in support of the conclusion $\alpha_1 < 0$ should be interpreted by means of some other theory.

THEOREM (Serrin, Dunn, and Fosdick). *Let a homogeneous incompressible fluid of grade 2 for which $\mu > 0, \alpha_1 \geq 0, \alpha_1 + \alpha_2 = 0$ undergo a flow filling a vessel, to the walls of which it adheres. Then*

$$0 \leq E(t) \leq E(0)e^{-t/\tau}, \tau = \frac{\alpha_1 + \rho C_P}{2\mu}. \tag{7.6-18}$$

If $\alpha_1 > 0$, then

$$E(0)e^{-t/\tau'} \leq E(t), \tau' = \frac{\alpha_1}{2\mu}. \tag{7.6-19}$$

PROOF From (9) to (1) we conclude that for any number τ,

$$\tau \dot{E} + E \leq (-2\mu\tau + \alpha_1 + \rho C_P) \int_V |\mathbf{G}|^2 dV. \tag{7.6-20}$$

From (3) and (4) we see that for any number τ',

$$\tau'\dot{E} + E \geq (-\mu\tau' + (1/2)\alpha_1) \int_V |\mathbf{A}_1|^2 dV. \qquad (7.6\text{-}21)$$

The choices of τ and τ' given by $(18)_4$ and $(19)_2$ annul the respective right-hand sides.

If $\alpha_1 = 0$, (18) reduces to one of Serrin's estimates for flows of Navier-Stokes fluids, which we shall encounter in Section 8.13. There is a difference. For the Navier-Stokes theory, the time τ for a given fluid is determined by the shape of the vessel. For the fluid of grade 2, that time is the sum of the Navier-Stokes time and a time characteristic of the fluid and independent of the vessel. If $\alpha_1 > 0$, (19) shows that K and the mean values of $|\mathbf{A}_1|^2$ and $|\mathbf{G}|^2$ tend to 0 as $t \to \infty$. As $\alpha_1 \to 0$, (19) reduces to the trivial statement $E(t) \geq 0$. Both the presence of a nonconstant, decreasing lower bound when $\alpha_1 > 0$ and the corresponding increase of τ in the upper bound given by (19) indicate that the corresponding nonlinear response has a stabilizing, regularizing effect such as Ting observed in his study of lineal flows.

The marked contrast in behavior effected by the two possible signs of α_1 may perhaps merely illustrate the difference between quadratic and linear approximations over a large interval. By taking ε small enough, we can approximate $x(1 + \varepsilon \sin x)$ with arbitrarily large accuaracy on $(-\infty, +\infty)$ by x, but the quadratic approximation $x(1 + \varepsilon x)$ becomes very bad for large $|x|$, no matter how small ε is, the sign of which dominates its behavior at ∞. Perhaps the fluid of grade 2 affords a possible model of real fluids only for short periods of time—not exactly the character of a good model. That, indeed, is all that the position established for it by the theory of retardation suggests.

Further theorems of stability and instability for fluids of grade 2 have been found.[12]

Recently, there have been several papers concerning the uniqueness of the solution to the equations governing the flow of fluids of grade two. Cioranescu and El Hacene[13] proved that there is a unique weak solution, for all time, for the initial-boundary value problem, if attention is restricted to plane flows in bounded domains. They also proved local existence (in time) of solutions for the full three-dimensional problem. Galdi, Grobbelaar and Sauer[14] showed that there is a unique

[12] J.E. Dunn and R.L. Fosdick, "Thermodynamics, stability, and boundedness of fluids of complexity 2 and fluids of second grade," *Archive for Rational Mechanics and Analysis* 56 (1974): 191–252; R.L. Fosdick and K.R. Rajagopal, "Anomalous features in the model of second-order fluids," *Archive for Rational Mechanics and Analysis* 70 (1979): 145–52.

[13] Cioranescu, D. and El Hacene, O., "Existence and uniqueness for fluids of second grade," in *Research Notes in Mathematics*, vol. 109 (Boston: Pitman, 1984), 178–197.

[14] Galdi, G.P., Grobbelaar-Van Dalsen, M., and Sauer, N., "Existence and uniqueness of classical solutions for the equations of motion of second grade fluids," *Arch. Rational Mech. Anal.* 124 (1993): 221.217.

classical solution (smooth solution), for short times to the initial-boundary value problem, in bounded domains. However, if the initial data are sufficiently small and the coefficient α_1 is a sufficiently large positive number, they showed that there is a unique classical solution for all time in bounded domains, irrespective of the value of $(\alpha_1 + \alpha_2)$. When $\alpha_1 = 0$, the model reduces to the Reiner-Rivlin model and the problem is mathematically more daunting than that for a fluid of grade two. Amaan[15] has recently proved that there is a unique solution to the initial-boundary value problem.

General References

[1.] Truesdell, C. "The meaning of viscometry in fluid mechanics." *Annual Reviews of Fluid Mechanics* 6 (1974): 111–146.

[2.] Truesdell, C. "Second-order effects in the mechanics of materials." Proceedings of the International Symposium of Second-Order Effects in Elasticity, Plasticity, and Fluid Dynamics. Haifa, Israel. April 21–29, 1962.

[3.] Truesdell, C. and Noll, W. *The Non-Linear Field Theories of Mechanics, Handbuch der Physik* 3_3: Berlin, Heidelberg, and New York, Springer-Verlag, 1965.

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8

Navier-Stokes Fluids

8.1 Purpose and Plan of Chapters 8 and 9

Classical hydrodynamics and aerodynamics concern fluids having linear viscosity or none at all. We defined these fluids in Chapter 4; in Chapters 5 and 7 we referred to them many times as special instances. Thus, in one sense, we have studied them already, but now we consider some of their properties that seem, as yet at least, peculiar to them, not to be understood easily as being merely the most special examples of more general ideas and flows.

For convenience we recall here the basic equations and conditions set forth in Chapter 4. Using the equation numbers provided, students should review the appropriate parts of Chapter 4 before reading this chapter.

1. *Eulerian compressible fluid*, also called *constrained elastic fluid*, *inviscid compressible fluid*, *ideal compressible fluid*:

$$\mathbf{T} = -p(\rho)\mathbf{1}. \tag{4.4-4}$$

2 *Eulerian incompressible fluid*, also called *incompressible elastic fluid*, *inviscid incompressible fluid*, *ideal incompressible fluid*:

$$\mathbf{T} = -p\mathbf{1}. \tag{4.4-13}$$

3. *Incompressible Navier-Stokes fluid*, also called *incompressible linearly viscous fluid*:

$$\mathbf{T} = -p\mathbf{1} + 2\mu\mathbf{D}. \tag{4.4-14}$$

The corresponding theories are called loosely the "Euler theory" and the "Navier-Stokes theory." In the last two equations the field p is not a given function of ρ and $\mathbf{D}$ but rather must be determined, as we see in examples, by the principle of linear momentum and suitable initial conditions and boundary conditions.

EXERCISE 8.1.1

Show that the condition $\mu > 0$ is equivalent to the statement that in a nonrigid simple shearing (2.2-13) the shear stress always drags the sheared planes. It is also equivalent to the statement that the stress power $\mathbf{T} \cdot \mathbf{G}$ is nonnegative in all nonrigid motions subject to stress that is not hydrostatic.

In Chapter 8 we study flows of incompressible Navier-Stokes fluids. Chapter 9 is devoted to Euler fluids. In conformity with what is stated at the end of Chapter 4 and in Section 5.3, the bodies of Navier-Stokes fluid we consider in this chapter are always incompressible and homogeneous; further, we *always assume that* $\mu > 0$ except for occasional references to statements that remain valid when $\mu = 0$ and hence provide information about a corresponding Euler fluid.

Substituting the constitutive relation $(4.1\text{-}7)_1$ into (3.8-4), we obtain the following *equation of motion*:

$$\ddot{\mathbf{x}} = \nu\Delta\dot{\mathbf{x}} = -\operatorname{grad}\varphi + 2\nu \operatorname{div}\mathbf{W}; \tag{8.1-1}$$

the constant ν, called the *kinematic viscosity*, is defined as follows:

$$\nu := \frac{\mu}{\rho}. \tag{8.1-2}$$

If $\nu > 0$, (1) is called *Navier's dynamical equation* for viscous, incompressible fluids; if $\nu = 0$, (1) is *Euler's dynamical equation* for ideal, incompressible fluids.

We see at once that if $\operatorname{div}\mathbf{W} = \mathbf{0}$, then, whatever the value of ν, (1) reduces to Euler's dynamical equation. In particular, if an isochoric flow of uniform spin (*e.g.*, an irrotational flow or a rigid motion) is compatible with a certain body force $\mathbf{b}$ according to the Euler theory, it is compatible with $\mathbf{b}$ according to the Navier-Stokes theory, as well, and with the same pressure field to within an arbitrary function of t only. This fact is not of much use, since in the Navier-Stokes theory it is customary to suppose that the fluid body adheres to the boundaries it touches. In particular, at a stationary boundary we apply the condition

$$\dot{\mathbf{x}} = \mathbf{0}. \tag{1.8-8}$$

This condition is generally incompatible with isochoric, irrotational flow. Indeed, an objective of the Navier-Stokes theory is to render the spin field of a fluid motion determinate from the dynamical equation and the conditions on boundaries and at ∞ and thus to account for the creation, diffusion, and destruction of vorticity and for the lift and drag that a body of incompressible fluid in steady flow may exert upon obstacles submerged in it. In many applications, a Navier-Stokes flow corresponding to a stationary boundary differs from a corresponding isochoric irrotational flow only in an adjacent, slender domain, which is called a *boundary layer*. In Section 7.4 we saw an example of this phenomenon for a fluid of grade 2. Below we shall study further examples for the Navier-Stokes fluid.

At this point students would do well to familiarize themselves with the magisterial treatise of Berker, cited at the end of this chapter. In it will be found many and various solutions according to the Navier-Stokes theory, explained in detail, provided with full and accurate references upon some of which the following text will draw, and often illustrated by diagrams. In this chapter and the following one we shall consider solutions of particular clarity and interest; some of these are to be found in Berker's treatise, while others were discovered after that treatise appeared. Students will learn that the Navier-Stokes equations, spiny though they are, do have some explicit, precise solutions of interest and value; will taste and feel some of the most famous of these and also some of the most recent; and will encounter some of the qualitative theorems that can be proved with a beginner's mathematical tools.

8.2 Condition of Compatibility

Flows of the Navier-Stokes fluid have to meet certain compatability requirements. To desire them we shall start with the equation of motion.

$$\ddot{\mathbf{x}} = -\operatorname{grad}\varphi + 2\nu \operatorname{div}\mathbf{W}. \tag{8.2-1}$$

We can consider, with no loss of generality, only flows that are affected by surface tractions alone in addition to kinematic boundary conditions. That we can do so is a consequence of a result due to Euler.[1] While in an Eulerian fluid those tractions are normal pressures, a body of Navier-Stokes fluid exerts tangential forces upon bounding surfaces. A stationary wall balances the drag exerted on it and in doing so forces the fluid at each point on the wall to spin about some tangential axis. The traction may be read off from Berker's theorem $(6.1\text{-}25)^1$: $\mathbf{t} = -p\mathbf{n} + 2\mu\mathbf{W}\mathbf{n}$; that is, the normal pressure on a wall equals the mean pressure, and the tangential traction equals the product μ and the vorticity vector rotated through a right angle counterclockwise about the outer normal vector of the wall.

A glance at (1) shows that flows of a viscous fluid cannot generally be determined from kinematical considerations alone. A solution will generally depend upon ν as a parameter, and the velocity fields possible for a fluid whose kinematic viscosity is ν_1 will not generally be possible for a fluid whose kinematic viscosity is ν_2. Indeed, the vorticity equation $(3.8\text{-}5)_2$ now reduces to

$$\dot{\mathbf{W}} + \mathbf{DW} + \mathbf{WD} = 2\nu \operatorname{skw}\operatorname{grad}\operatorname{div}\mathbf{W}; \tag{8.2-2}$$

students will recall that

$$\mathbf{W}_{\mathbf{a}} = \dot{\mathbf{W}} + \mathbf{DW} + \mathbf{WD}. \tag{2.2-30}$$

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[1] C. Truesdell, *A First Course in Rational Continuum Mechanics*, (New York, Academic Press: 1991).

The *condition of compatibility* (2) is necessary and sufficient that the velocity field $\dot{\mathbf{x}}$ whence $\mathbf{W}$ and $\mathbf{D}$ are possible for a fluid of kinematic viscosity ν. If it is satisfied, a function φ such as to satisfy (1) exists locally and is unique to within a function of t only. With φ determined and ϖ prescribed, (3.7-5) determines $p\nu$ to within a function of t only.

8.3 Some Bernoullian Theorems

A *Bernoullian theorem* is an explicit determination of the pressure from a velocity field known to be compatible with an equation of motion. We now consider some Bernoullian theorems. The determination (6.6-4) for universal solutions is itself a Bernoullian theorem, and we shall first note its application to irrotational flows. For them, div $\mathbf{A}_1 = 0$, so $P_{\mathfrak{f}} = 0$; hence

$$P_{\mathfrak{a}} = \varphi. \tag{8.3-1}$$

EXERCISE 8.3.1 (Euler)
Show that in an irrotational flow,

$$P_{\mathfrak{a}} = P_{\mathrm{v}}' - \frac{1}{2}|\dot{\mathbf{x}}|^2 = P_{\mathrm{v}}' - \frac{1}{2}|\operatorname{grad} P_{\mathrm{v}}|^2. \tag{8.3-2}$$

Eliminating $P_{\mathfrak{a}}$ from (1) and (2) yields

$$\varphi + \frac{1}{2}|\dot{\mathbf{x}}|^2 = P_{\mathrm{v}}'. \tag{8.3-3}$$

This statement, due to Euler, is often called Bernoulli's theorem for irrotational flow. The name comes from its resemblance to theorems of hydraulics discovered by Daniel Bernoulli and John Bernoulli. The fluid undergoing the flow may be viscous or inviscid. As we mentioned earlier, irrotational flows are generally incompatible with the condition of adherence to a stationary wall and thus, while possible for viscous fluids, are rarely appropriate to them except in regions far from boundaries.

There is a Bernoullian theorem for a solution of Navier's dynamical equation (8.1-1) that need not be universal, though for it the spin of $\mathbf{W}$ must be steady: $\mathbf{W}' = \mathbf{0}$. To obtain it, we need to use the kinematical theorem stated in the following exercise.

EXERCISE 8.3.2

If $\mathbf{W}' = \mathbf{0}$ in a region, show that the local acceleration has a potential there, say $\dot{\mathbf{x}}' = \operatorname{grad} Q$, and hence

$$\ddot{\mathbf{x}} = \operatorname{grad}\left(Q + \frac{1}{2}|\dot{\mathbf{x}}|^2\right) + \mathbf{W}\dot{\mathbf{x}}. \tag{8.3-4}$$

The Bernoullian theorem we now present by using (4) is associated with the vector lines of the normals, assumed Lipschitz-continuous, of the planes defined by the vectors $\mathbf{W}\dot{\mathbf{x}}$ and $\operatorname{div}\mathbf{W}$. These curves are independent of ν; to obtain them we need only integrate the ordinary differential equation $\frac{d\mathbf{x}}{ds} = \mathbf{W}\dot{\mathbf{x}} \times \operatorname{div}\mathbf{W}$.

EXERCISE 8.3.3 (Truesdell)

Let $\dot{\mathbf{x}}$ have a steady field of spin and be a solution of Navier's equation, and let C be a curve that is normal to both $\mathbf{W}\dot{\mathbf{x}}$ and $\operatorname{div}\mathbf{W}$. Show that there is a function k_C of time alone such that on C

$$\varphi + Q + \frac{1}{2}|\dot{\mathbf{x}}|^2 = k_C. \tag{8.3-5}$$

Simpler conclusions hold for flows in two degenerate classes:

1. $\mathbf{W}\dot{\mathbf{x}} \neq 0$ and $\nu \operatorname{div}\mathbf{W} \parallel \mathbf{W}\dot{\mathbf{x}}$. (8.3-6)
2. $\mathbf{W}\dot{\mathbf{x}} = 0$.

For steady flow, all the foregoing theorems become simpler. In steady irrotational flow we may put 0 for P'_{v} in (3), so obtaining

$$\varphi + \frac{1}{2}|\dot{\mathbf{x}}|^2 = \text{const. (throughout)}, \tag{8.3-7}$$

an equation having just the same form as the Bernoullian theorem of steady hydraulics. Likewise, for steady flow, the curves C are stationary, and we may put 0 for Q in (5) to obtain

$$\varphi + \frac{1}{2}|\dot{\mathbf{x}}|^2 = \text{const. on each curve } C. \tag{8.3-8}$$

These exercises tell us that φ must decrease when the speed increases, and vice versa; also the constants may be regarded as maximum values of φ, appropriate to stagnation points. The only difference is that while for an irrotational flow the increases and decreases refer to the whole region on which the flow is defined, for a rotational flow they refer in general only to the several curves C. In both conclusions *φ is determined by the velocity field alone*, to within arbitrary constants. Thus, if we have a steady solution $\dot{\mathbf{x}}$ of the compatibility condition (8.2-2) for some value of ν, and if ϖ is prescribed, the Bernoullian theorem delivers the pressure field p

that provides the solution of the dynamical equation corresponding with the known velocity field $\dot{\mathbf{x}}$. In this sense we may regard the compatibility condition as the only equation that needs to be solved to find steady flows of linearly viscous fluids.

In a formal sense, solution of the Navier-Stokes equations is reduced to a purely kinematical problem for each value of ν. On the other hand, it is not easy to take account of boundary conditions and initial conditions, since these must be expressed in terms of the vorticity alone if we are to determine $\mathbf{W}$ from (8.2-2).

8.4 Dynamical Similarity

The name "kinematic viscosity" is motivated by the condition of compatibility (8.2-2), for ν is a factor of proportionality between two kinematic fields. The physical dimensions of μ are those of mass divided by the product of length and time; the physical dimensions of ν, which are those of squared length divided by time, are purely kinematical.

The appearance of a dimension-bearing modulus suggests that by recourse to dimensionless variables we may remove from any given problem its apparent dependence upon the choice of units, which are always arbitrary. If we let some particular length and speed be L and V, respectively, we can introduce as follows dimensionless fields $\dot{\mathbf{x}}^0$ and p^0 corresponding with $\dot{\mathbf{x}}$ and p : $\dot{\mathbf{x}}^0 := V^{-1}\dot{\mathbf{x}}$, $p^0 := L^{-1}p$. Then in an obvious notation $\mathbf{W} = L^{-1}V\mathbf{W}^0$, $\mathbf{W}_a = L^{-2}\mathbf{W}^0_{\mathbf{a}}$, grad $= L^{-1}$ grad, and so on, so (8.2-2) becomes the dimensionless statement

$$\mathbf{W}^0_{\mathbf{a}} = \frac{2}{\mathcal{R}} \operatorname{skw} \operatorname{grad}^0 \operatorname{div}^0 \ \mathbf{W}^0, \tag{8.4-1}$$

in which

$$\mathcal{R} := \frac{VL}{\nu} \tag{8.4-2}$$

The dimensionless number $\mathcal{R}$ is called "the" *Reynolds number*, although, as the definition indicates, for the same fluid there are infinitely many such numbers, one for each choice of the product VL. We have shown that a single velocity field satisfying the compatibility condition (8.2-2) gives rise to infinitely many others, which are obtained from it by multiplying all lengths by a constant L and all velocities by a constant V, provided the kinematic viscosity is such that $\mathcal{R}$ has the same value as for the original flow. $\mathcal{R}^{-1}$ is often called the *Ekman number*.

In this way are compared not only flows of one and the same fluid, but also flows of fluids different from each other in that they have different kinematic viscosities. Such comparison is of manifold use in experiments using models.

A transformation of this kind is called a *similitude*; the rule specifying the similitudes is called a *law of scaling* or a *law of dynamical similarity*. Only the simplest theories of materials allow a nontrivial law of scaling. The example provided by the Navier-Stokes theory is the best known and most useful of these laws.

In any particular configuration, the constants V and L are suggested naturally. For example, in a flow of a fluid body that surrounds a sphere and has a uniform velocity at infinity, we take V as the magnitude of that velocity, L as the radius of that sphere. The scaling law then permits us to compare flows of fluids of different kinematic viscosities past spheres of different radii and with different speeds at infinity. The boundary condition of adherence, which is the most usual in applications of the Navier-Stokes theory, is preserved under a similitude; if the initial velocity field is prescribed, it is simply multiplied by V in the similitude.

Even the phrasing of the last statement suggests a common and usually not justified converse to what has been proved. We have shown only that a transformation such as to leave $\mathcal{R}$ invariant transforms one solution of (8.3-4) into another one. Indeed, (1) shows that a velocity field compatible with the Navier-Stokes theory depends upon $\mathcal{R}$ as a parameter; nonetheless, it does not show that only one velocity field satisfying certain boundary conditions corresponds to each value of $\mathcal{R}$. Without such a uniqueness theorem, the common interpretation of the scaling law is uncertain.

The theory of scaling just discussed is incomplete in that it considers only (8.2-2), the condition of compatibility. If we refer to the dynamical equation (8.1-1), we see that further conditions must be satisfied. Under a transformation that preserves compatibility, the function φ associated with a solution of (8.1-1) is divided by V^2. Since $\varphi := pU + \varpi$, this fact is easiest to interpret if we regard pU and ϖ as being individually divided by V^2. If ϖ is assigned rather than disposable and if $\varpi \neq$ const., this rule amounts to a restriction upon L, V, and a scaling constant for the time. For example, if $\varpi = gh$, where g is a constant acceleration such as that of gravity and h is the height above some horizontal plane, then the statement is most easily expressed in terms of the *Reech number*, often called the Froude number:

$$\mathcal{F} = \frac{gL}{V^2}. \tag{8.4-3}$$

For one solution to generate another through the simple transformations described, it is necessary that $\mathcal{F}$ have the same value for both. For example, for an experiment upon a flow of a Navier-Stokes fluid past a ship model to be able through mere multiplication of velocities and lengths by constant factors to provide results appropriate to the flow of a ship of similar form, the ratio L/V^2 must be the same for both flows. This statement is Reech's *law of similitude*. It presumes that g is fixed. A broader class of similitudes is obtained by allowing different values of g, as do occur at different altitudes or through the action of fields of force other than uniform gravity.

Likewise, the ratio pU/V^2 must have the same value for both flows, and here p is not a constant but a function of place and time. The natural way to interpret the rule of similitude here is to regard it as delivering the pressure field of the transformed flow. Namely, at corresponding points the ratio pU for the similar flow is obtained by multiplying that ratio for the original flow by $V/|\dot{\mathbf{x}}|^2$. Even more simply, we may say that the pressure field for the similar flow may be obtained from it just as the given pressure field was obtained from the given flow. When a particular pressure p_0 is singled out by the statement of the problem, we can express this same rule by saying that the similar flows must have a common "Euler number" $p_0\nu/V^2$.

8.5 Viscometry

Students should already have mastered the Navier-Stokes theory of viscometry, which was explained in Section 5.4 and illustrated several times thereafter, and the simpler aspects of the Navier-Stokes theory of flow through straight pipes, which was presented in step 1 in Section 6.4. It is an easy and instructive exercise to define a Reynolds number for each of the flows exhibited and to discuss the effect of that number upon the nature of the flow. For example, if in a channel flow we take V as the average speed and L as the breadth of the channel, we find that $D = VL$, so we can write the conclusion (5.4-19) in the form $(D/\nu) = \mathcal{R}$, while if in flow through a pipe we take V as the average speed and L as the radius R of the pipe, we can write (5.4-41) as $D/\pi R\nu = \mathcal{R}$.

8.6 Flow against and along a Plate

In Chapter 7 we presented several flows of fluids of grade 2, and we discussed the Navier-Stokes flows to which they reduce when $\alpha_1 = \alpha_2 = 0$. We now discuss some other flows that can be undergone by an incompressible Navier-Stokes fluid. There are many more. For a plane Navier-Stokes fluid the partial differential equation satisfied by the stream function can be read off from (7.3-3) and (7.3-2):

$$\nabla\Delta q \cdot (\Delta q)^{\perp} = \nu\Delta\Delta q. \tag{8.6-1}$$

In this section we shall use Cartesian coordinates x_1, x_2, x_3, the last to lie in a direction normal to the plane considered. An evident solution of (1) is $q = Ax_1x_2$, which corresponds with the velocity field $\dot{x}_1 = Ax_1, \dot{x}_2 = -Ax_2$, an isochoric, irrotational flow of fourfold symmetry with stagnational points on the line through the origin and perpendicular to the plane of flow. Valid for all values of ν, this solution does not show any effect of viscosity. Solutions that do so may be sought in the following broader class of plane isochoric flows, set up as a basis for a semiinverse approach:

$$\dot{x}_1 = x_1 f'(x_2),\ \dot{x}_2 = -f(x_2),\ \dot{x}_3 = 0. \tag{8.6-2}$$

According to Schlichting,[2] Hiemenz was the first to analyze these fields. For them, (1) simplifies greatly and after one integration delivers the statement

$$f'^2 - ff'' - \nu f''' = A^2 = \text{const.} \tag{8.6-3}$$

The solution $f = Ax_2$ is obvious; it is independent of ν and is the only irrotational solution.

EXERCISE 8.6.1
Show that

$$p = -1/2\rho(f^2 + A^2 x_1^2) - \mu f'' + \text{const.} \tag{8.6-4}$$

If $f = Ax_2$, this statement reduces to the Bernoulli equation for irrotational flow.

Henceforth, assuming that $\nu > 0$ and $A > 0$, we shall use dimensionless variables defined as follows:

$$\eta := \sqrt{\frac{A}{\nu}} x_2,\ F(\eta) := \frac{f(x_2)}{\sqrt{A\nu}}, \tag{8.6-5}$$

in terms of which (3) becomes

$$F''' + FF'' - F'^2 + 1 = 0. \tag{8.6-6}$$

The solution $F = \eta$ now represents the irrotational flow. Adherence to the plane $x_2 = 0$ is expressed by the requirements

$$F(0) = 0,\ F'(0) = 0; \tag{8.6-7}$$

the first of these is satisfied by the irrotational flow, while the second is violated by it. When both are satisfied, the line of stagnation points is replaced by a plane of them. There is a solution of (6) that approximates the irrotational flow at great distances from the stagnation plane:

$$F' \to 1 \text{ as } \eta \to \infty. \tag{8.6-8}$$

For such a solution we see from (6) that

$$F''' + FF'' \to 0 \text{ as } \eta \to \infty. \tag{8.6-9}$$

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[2] H. Schlichting summarizes and cites researches on this subject in vol. 9 of his *Boundary Layer Theory* (New York: McGraw-Hill, 1979). He reproduces there a table of values of F, f', and F'' calculated numerically by Howarth as functions of η. Also see K. Hiemenz, "Die Grenzschicht an einem in den gleichförmingen Flussigkeitsstorm in eintauchten gerader Kreiszylinder."

EXERCISE 8.6.2

Verify that the conclusion (9) is equivalent to the statement that the pressure gradient in the direction normal to the plane approaches at infinity that of the corresponding irrotational flow.

From (2) it is easy to conclude that

$$\frac{(\dot{x}_1)_{\text{irr}} - \dot{x}_1}{(\dot{x}_1)_{\text{irr}}} = 1 - F', \quad \frac{\dot{x}_2 - (\dot{x}_2)_{\text{irr}}}{(\dot{x}_1)_{\text{irr}}} = \sqrt{\frac{\nu}{A}}\frac{\eta - F}{x_1}, \tag{8.6-10}$$

in which subscript irr designates components of the corresponding irrotational flow. The assigned condition (8) makes $\dot{x}_1$ tend to $(\dot{x})_{\text{irr}}$ as $\eta \to \infty$; for a fixed η, from (6) we see that the ratio on the left-hand side of $(10)_1$ tends to 0 also if $\nu \to 0$ while A and η are held fixed. The ratio on the left-hand side of $(10)_2$ tends to 0 if $\nu \to 0$ while A, η, and x_1 are held fixed. Numerical calculation by Howarth indicates that as $\eta \to \infty$, $F' \to 1$ monotonically from below, and $F' > 0.99$ when $\eta > 2.4$. Thus

$$\frac{(\dot{x}_1)_{\text{irr}} - \dot{x}_1}{(\dot{x}_1)_{irr}} < 0.01 \text{ if } x_2 > 2.4\sqrt{\frac{\nu}{A}}. \tag{8.6-11}$$

This statement indicates that there is a boundary layer whose "shearing thickness" is of order $\sqrt{\nu/A}$, independent of the distance along the plate. Numerical calculations suggest also that $F \sim \eta - 0.65$ as $\eta \to \infty$, the sign $\sim$ indicating "asymptotic to"; thus $\dot{x}_2 \sim -\sqrt{A\nu}(\eta - 0.65)$. It seems also that the left-hand side of (10) is bounded by $0.65\sqrt{\nu/A}/x_1$ as $\eta \to \infty$. In other words, if we define as follows a "displacement thickness" δ_{x_2};

$$\delta_{x_2} := 0.65\sqrt{\nu/A}, \tag{8.6-12}$$

then $\dot{x}_2 \sim -A(x_2 - \delta_{x_2})$; that is, the irrotational flow is pushed away from the stagnation plane by approximately the distance δ_{x_2}.

8.7 Flow about a Rotating Circular Cylinder with Suction

An elegant plane solution due to Hamel corresponds to an infinite mass of fluid driven by an infinitely long circular cylinder of radius r_0, rotating at constant angular speed and providing radial suction upon its surface.

Again appealing to a semiinverse approach, we assume that the flow is obtained by superposing a simple vortex (2.2-14) and a line sink located upon the axis of the vortex. In contravariant cylindrical components

$$\dot{r} = u(r), \ \dot{\theta} = \omega(r), \ \dot{z} = 0. \tag{8.7-1}$$

The condition $(2.5\text{-}8)_1$ for isochoric motion becomes

$$\frac{1}{r}\frac{d}{dr}(ru) = 0. \tag{8.7-2}$$

Hence, $ru = \text{const.}$; equivalently

$$u = -V\frac{r_0}{r}, \tag{8.7-3}$$

in which V denotes the speed at which fluid is sucked into the surface of the cylinder $r = r_0$. The streamlines are spiral: $\theta = -r^2\omega(r)/(Vr_0) + \text{const}$. The vorticity is that of a simple vortex; hence its magnitude w is given by $rw = (r^2\omega)'$, in which, as henceforth in this subsection, the prime denotes differentiation with respect to r; it follows that the circulation of the circle $r = \text{const.}$ is $2\pi r^2\omega(r)$. A straightforward calculations delivers the θ-component of the vorticity equation (8.2-2) for this flow: $\dot{\mathbf{W}} + \mathbf{DW} - \mathbf{WD} = 2\nu \operatorname{skw} \operatorname{grad} \operatorname{div} \mathbf{W}$,

$$-\mathcal{R}w' = (rw')', \quad \mathcal{R} := Vr_0/\nu, \tag{8.7-4}$$

in which the choice of the Reynolds number $\mathcal{R}$ is a natural instance of the general idea expressed by (8.4-2). The solution of (4) is

$$w - 2\Omega = (w_0 - 2\Omega)\left(\frac{r_0}{r}\right)^{\mathcal{R}}, \tag{8.7-5}$$

where w_0 is the spin of the fluid on the cylinder $r = r_0$, and Ω is the angular speed of the fluid mass at ∞ : $w \to 2\Omega$ as $r \to \infty$. If $\mathcal{R}$ is large, the vorticity induced by the rotation of the cylinder is confined to a narrow boundary layer whose thickness decreases in proportion to $(r_0/r)^{\mathcal{R}}$ as $r \to \infty$. The thickness δ of the layer in which $(w - 2\Omega)/(w_0 - 2\Omega) > \varepsilon$ is given by $\delta/r_0 = \varepsilon^{-1/\mathcal{R}}$.

EXERCISE 8.7.1

By integrating the relation $rw = (r^2\omega)'$ show that the circulation of the cylinder $r = \text{const.}$ is given by:

$$2\pi r^2(\omega - \Omega) = \begin{cases} \frac{2\pi(w_0-2\Omega)r^2}{2-\mathcal{R}}\left(\frac{r_0}{r}\right)^{\mathcal{R}} + 2\pi r_0^2\left(\omega_0 + \frac{\mathcal{R}\Omega - w_0}{2-\mathcal{R}}\right) \text{ if } \mathcal{R} > 2, \\ 2\pi(w_0 - 2\Omega)r_0^2 \log \frac{r}{r_0} + 2\pi r_0^2(\omega_0 - \Omega) \text{ if } \mathcal{R} \le 2, \end{cases} \tag{8.7-6}$$

in which $\omega_0 := \omega(r_0)$. In particular, the circulation at $r = r_0$ is $2\pi r_0^2(\omega_0 - \Omega)$. Here we see a great difference between the outcomes for large $\mathcal{R}$ and small $\mathcal{R}$. If $\mathcal{R} \le 2$, then the circulation $2\pi r^2(\omega - \Omega)$ induced by the rotating cylinder tends in general to ∞ as $r \to \infty$. The only solution having finite circulation at ∞ is the rigid motion $\omega = \omega_0 = \Omega$, $\mathrm{w} = \mathrm{w}_0 = 2\Omega$. If $\mathcal{R} > 2$, then $2\pi r^2(\omega - \Omega)$ tends to the limit C_∞:

$$C_\infty = 2\pi r_0^2\left(\omega_0 + \frac{\mathcal{R}\Omega - \mathrm{w}_0}{\mathcal{R} - 2}\right). \tag{8.7-7}$$

The unique solution is thus determined by the assigned constants R, ω_0, Ω, and C_∞, in terms of which we may calculate w_0, the spin of the fluid adhering to the cylinder:

$$\mathrm{w}_0 - 2\Omega = (\mathcal{R} - 2)\left[\frac{C_\infty}{2\pi r_0^2} - (\omega_0 - \Omega)\right] \tag{8.7-8}$$

While, as we saw, the decay of vorticity at ∞ is proportional to $(r/r_0)^{\mathcal{R}}$, that for circulation is proportional to $(r/r_0)^{\mathcal{R}-2}$.

8.8 Flow between Intersecting Planes

Jeffreys and Hamel analyzed flows of a Navier-Stokes fluid within an infinite wedge, to whose sides the fluid adheres and whose vortex is a line source or a line sink. They showed that radial flow, either outflow or inflow, was possible for all angles of the wedge.

We suppose the source or sink to be the axis of a cylindrical coordinate system r, θ, z, so the radial planes are represented by θ = const. We shall locate the bounding planes as $\theta = \pm - 1/2\alpha$, so the angle of the wedge is α; we shall suppose that $0 < \alpha \leq 2\pi$. A stream function of the form

$$q = F(\theta) \tag{8.8-1}$$

corresponds to radial flow; the discharge D through the part of a right cylinder r = const. of unit height that is cut off by the planes $\theta = \theta_1$ and $\theta = \theta_2$ is given by $F(\theta_2) - F(\theta_1)$. Thus for the discharge D through the entire wedge we have

$$D = F\left(\frac{1}{2}\alpha\right) - F\left(-\frac{1}{2}\alpha\right). \tag{8.8-2}$$

The velocity field corresponding to (1) is

$$\dot{r} = \frac{F'(\theta)}{r}, \ \dot{\theta} = 0, \ \dot{z} = 0. \tag{8.8-3}$$

Consequently a positive F' provides outflow, a negative F', inflow. Because the fluid adheres to the planes $\theta = \pm\frac{\alpha}{2}$,

$$F'\left(\pm\frac{\alpha}{2}\right) = 0. \tag{8.8-4}$$

With the conventions and notations we have adopted, $D > 0$ if the axial line is a source, while $D < 0$ if that line is a sink.

Substituting (1) into (8.6-1) yields the following differential equation for F:

$$\nu(F^{\text{iv}} + 4F'') + 2F'F'' = 0, \tag{8.8-5}$$

the integral of which is

$$F''' + 4F' + \frac{1}{\nu}F'^2 = C, \tag{8.8-6}$$

where C is an arbitrary constant.

We now introduce dimensionless variables,

$$\mathcal{R} \equiv \frac{|D|}{\nu}, \quad U \equiv \frac{F'}{|D|}, \tag{8.8-7}$$

the former of which we take for the Reynolds number of the flow. In terms of them, (6), (4), and (2) become, respectively,

$$U'' + 4U' + \mathcal{R}U^2 - E = 0, \tag{8.8-8}$$

$$U\left(\pm\frac{1}{2}\alpha\right) = 0, \tag{8.8-9}$$

$$\int_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} U(\theta)d\theta = \varepsilon, \tag{8.8-10}$$

in which $E \equiv C/|D|$ and $\varepsilon = +1$ for a source, $\varepsilon = -1$ for a sink. The differential equation (8) is easily integrated once:

$$\begin{aligned} U'^2 &= \frac{2}{3}\mathcal{R}P(U), \\ P(U) &= -U^3 - \frac{6}{\mathcal{R}}U^2 + E_1 U + E_2, \end{aligned} \tag{8.8-11}$$

where E_1 and E_2 are constants.

While the solution of (11) may be expressed in terms of elliptic functions, a direct, qualitative analysis is more enlightening. The outcome depends upon α and the three parameters $\mathcal{R}$, E_1, and E_2, and equivalently upon the three zeros of the polynomial P. For the details, consult Chapter 7 Section 11 of Berker's treatise cited at the end of the chapter. The nature of the solutions can be described roughly as follows. The possibility of a purely convergent or purely divergent flow depends upon both $\mathcal{R}$ and α. If $\mathcal{R}$ is small and $\frac{\alpha}{2}$ is small, unique inflow and unique outflow exist and have much the same character. If α is large and is precisely the period of U, then a flow partly convergent and partly divergent is possible. Then $U(\beta) = 0$ for some β in $(-\frac{1}{2}\alpha, \frac{1}{2}\alpha)$. The fluid is then at rest on the plane $\theta = \beta$, on one side of which it flows outward, on the other, inward. To ensure unique inflow or unique outflow we must add the requirement that F', shall not vanish in $(-\frac{\alpha}{2}, \frac{\alpha}{2})$. The distribution of speeds is like that in a flow in a straight channel (see (5.4-16)). For an inflow corresponding to large $\mathcal{R}$ the speeds, except in two layers adjacent to the bounding planes, are nearly uniform, as they would be for an Eulerian fluid. The layers are contained in wedges whose angles are of the order of $1/\mathcal{R}$; in each wedge the speed increases rapidly with the angle θ. In contrast, for a given opening α there is a limit R_{ult} such that no outflow exists if $R > R_{\text{ult}}$. It follows that if $R > R_{\text{ult}}$, the flow corresponding to a source must exhibit not only outflow but also one or more wedges of inflow.

8.9 Flow in a Round Jet

A three-dimensional analogue of the flow considered in the preceding subsection is a round jet produced by a point source. We approach flows of this kind by a semiinverse method, laying down for study the following covariant components of an axisymmetric, isochoric velocity field in a spherical coordinate system r, θ, ϕ:

$$\dot{r} = -\frac{\nu}{r \sin\theta} F'(\theta), \quad \dot{\theta} = \frac{\nu}{r^2 \sin\theta} F(\theta), \qquad \dot{\phi} = 0. \tag{8.9-1}$$

We attempt to determine the dimensionless function F to satisfy the vorticity equation (8.2-2). Setting $\eta := \cos\theta$ and $f(\eta) := F(\theta)$, we find that

$$2(1-\eta^2)f' + 4\eta f - f^2 = C_1\eta^2 + C_2\eta + C_3, \tag{8.9-2}$$

in which the prime now denotes differentiation with respect to η, while C_1, C_2, and C_3 are arbitrary constants. Berker states that this Riccati equation (2) is due to Slezkin and Landau: the latter associated to flow in a round jet, but there are difficulties in finding instances that satisfy the condition of adherence to a stationary boundary.

EXERCISE 8.9.1
The flow (1) is isochoric, and the streamlines in a plane through the polar axis are the curves $f =$ const. Visualize and sketch the streamlines corresponding to some instances of (1). Substituting (1) into (8.2-2) and integrating the outcome three times delivers (2).

EXERCISE 8.9.2
The substitution

$$f = \frac{-2(1-\eta^2)G'}{G} \tag{8.9-3}$$

in (2) delivers a hypergeometric differential equation for G.

Some particular solutions of (2) are easy to get. We first suppose that $C_1 = C_2 = C_3 = 0$. Then the solution of (2) is

$$f = \frac{2(1-\eta^2)}{a-\eta}, \qquad a = \text{const.} \tag{8.9-4}$$

EXERCISE 8.9.3 (Squire)
If $a > 1$, show that the streamlines $f = \text{const.}$ experience minima as they cross the line $\eta = 1/a$. The velocity does not vanish anywhere.

While Squire interpreted this solution as representing a round jet bounded by one of the cylindrical streamsheets up to its minimum cross-section, the condition of adherence is not satisfied on any such sheet; Squire stated that "a special frictional boundary condition" is satisfied on the walls of the nozzle. If the jet is free or to emerge into another body of fluid, a special boundary condition must be satisfied there, too.

Another instance is obtained by putting $C_2 = -2C_1$, $C_3 = C_1$. There are then two particular solutions of (2):

$$f = K(1-\eta), \quad K = -1 \pm \sqrt{1-C_1}. \tag{8.9-5}$$

Squire assumed that $C_1 > 0$; then $C_1 = 1 + 4b^2$, $b > 0$, and $(4)_2$ takes the form $-\frac{K}{2} = (1/2) \pm ib$. The theory of the Riccati equation then delivers the general solution of (2) for the conditions $C_3 = C_1 > 0$, $C_2 = -2C_1$:

$$f = -(1-\eta)\left(1 - 2b\frac{\sin\varphi - D\cos\varphi}{\cos\varphi + D\sin\varphi}\right), \quad \varphi := \log(1+\eta)^b \tag{8.9-6}$$

The constant of integration D may be determined by making the equatorial plane a streamsheet: $f(0) = 0$. Then $D = -(1/2)/b$, so

$$f = (4b^2+1)\frac{1-\eta}{2b\cot\varphi - 1}. \tag{8.9-7}$$

Squire interpreted this solution as appropriate to a round jet emerging from the point of intersection of the polar axis and the equatorial plane, but the velocity does not vanish on the equatorial plane. The streamsheets corresponding to motion up the polar axis do not have conical or cylindrical asymptotes.

Squire studied also the instance in which $C_1 = C_2 = 0$, $C_3 < 0$ so that $C_3 = 4(1-b^2)$, $b > 1$. There are then two particular solutions of (2), namely,

$$f = 2(\eta \pm b), \tag{8.9-8}$$

and from the theory of Ricatti equations, the general solution of (2) under the assumption laid down:

$$f = -2\frac{(1+\eta)^b(b-\eta) - D(1-\eta)^b(b+\eta)}{(1+\eta)^b + D(1-\eta)^b}, \tag{8.9-9}$$

in which D is a constant of integration. D may be determined by making the cone $\eta = \eta_0$ a streamsheet: $f(\eta_0) = 0$. Then

$$D = \left(\frac{1+\eta_0}{1-\eta_0}\right)^b \frac{b-\eta_0}{b+\eta_0}. \tag{8.9-10}$$

This class of solutions, if generously interpreted, includes various jets. For the details, which are interesting, and for other special solutions of (2), consult section 21 of Berker's treatise.

8.10 Swirling Flow between Rotating Plates

In a famous memoir, Von Karman introduced and studied axially symmetric flow induced in a body of Navier-Stokes fluid confined by a rotating plane and a stationary plate. Mathematicians subsequently have given more scrutiny to flows of this kind than to any other special class of solutions of the Navier-Stokes equations. Even so, some questions regarding the existence, uniqueness, and stability of the solutions remain unanswered. Recently Berker opened a new avenue by exhibiting a one-parameter family of flows induced by two parallel plates rotating with the same constant angular speed about a common axis. Although this class is defined by an axially symmetric statement, only one member of it is axially symmetric, and that one is a rigid motion. This discovery bears upon the stability of axially symmetric solutions in general. For details of the vast analytical and numerical study of these problems, consult the surveys by Parter[3] and Rajagopal.[4]

Here we shall merely set up the equations and notice some of their properties.

8.10.1 Axially symmetric solutions

The axially symmetric solutions are instances of the isochoric velocity fields whose contravariant components in cylindrical coordinates r, θ, z are

$$\dot{r} = (1/2)rh'(z), \quad \theta = (1/2)g(z), \quad \dot{z} = h(z). \tag{8.10-1}$$

The functions g and h are to be determined to provide a solution of the Navier-Stokes equations. The rotating plates are represented by the planes $z = \pm d$. If

[3] S.V. Parter, "On the swirling flow between rotating coaxial disks: A survey," in *Theory and Applications of Singular Perturbations*, edited by W. Eckhaus and E.M. de Jager, Lecture Notes in Mathematics no. 942, (Berlin, Heidelberg, and New York: Springer-Verlag: 1982), 258–80.

[4] K.R. Rajagopal, "Flow of viscoelastic fluids between rotating disks," *Theoretical and Computational Fluid Dynamics* (1992): 185–206.

we substitute (1) into (8.2-2), we obtain the following differential system in the interval $-d \le z \le d$:

$$\nu h^{iv} + hh''' + gg' = 0,$$
$$\nu g^{iv} + hg' - h'g = 0. \tag{8.10-2}$$

Adherence of the fluid to the rotating plates requires that

$$h(-d) = h(d) = h'(-d) = h'(d) = 0,$$
$$g(-d) = 2\Omega_{-d}, \quad g(d) = 2\Omega_d, \tag{8.10-3}$$

in which the constants Ω_d and Ω_{-d} are the angular speeds of the plates $z = d$ and $z = -d$ in their rotation about the axis $r = 0$. The statements (2) and (3) set the "V. Karman problem." While Von Kármán used the semiinverse representation (1) to study the flow due to a single rotating plate, the same representation also suffices for flow between two plates rotating about a common axis. We shall bear in mind that solutions g and h will usually depend upon ν, d, Ω_d, and Ω_{-d} as well as z.

If the two plates rotate at the same angular speed, then $\Omega_{-d} = \Omega_d$, and we notice at once the solution

$$h(z) \equiv 0, \quad g(z) = 2\Omega_{-d}, \tag{8.10-4}$$

which is a rigid motion, unaffected by viscosity. It is well known that this solution is "stable" and "isolated" with respect to the system (2), (3). "Isolated" means that in a neighborhood of (4), defined by a suitable topology, there is no other solution of (2) and (3). The solution is also "stable" because for no value of ν can a family of other solutions branch off from it, and also the linearized problem corresponding to (2) and (3) has no singularity at this solution.

We shall now show that the axially symmetric solutions are never isolated when a more general class of solutions is allowed from the start.

8.10.2 Solutions that are not axially symmetric

In the course of our study of the fluid of grade 2 we exhibited the flow (4.2-35), which represents a body of fluid confined by two parallel plates rotating with constant and equal angular speeds about an axis parallel to the z-axis and a distance ℓ from it. The corresponding Navier-Stokes flow is obtained by taking α_1 as 0, whence we conclude that $m^2 = n^2 = (1/2)\,\Omega/\nu$. The outcome is Berker's family of solutions depending on the parameter ℓ, none of which is axially symmetric except the rigid motion obtained by putting $\ell = 0$. That rigid motion is the unique solution of (2) and (3) when $\Omega_d = \Omega_{-d}$. The same solution is the outcome of taking the limit in (4.2-35) as $\ell \to 0$ when m and n are fixed. Thus the rigid solution, while isolated in the class of axially symmetric solutions, is not isolated in the broader class considered by Berker.

This fact suggested to Rajagopal that also when $\Omega_d = \Omega_{-d}$, an axially symmetric solution of the kind long studied by analysts might be imbedded in a family of solutions that do not exhibit such symmetry. To that end he introduced a broader class of isochoric flows for application of the semiinverse method:

$$\begin{aligned}\dot r &= (1/2)rh'(z) + k(z)\cos\theta - f(z)\sin\theta,\\ \dot\theta &= (1/2)g(z) - \frac{1}{r}k(z)\sin\theta - \frac{1}{r}f(z)\cos\theta,\\ \dot z &= h(z).\end{aligned} \tag{8.10-5}$$

If $h := 0$ and $g := 2\Omega = \text{const.}$, (5) reduces to the velocity field studied by Berker and considered here in Section 5.6 and Section 7.5, while if $f := 0$ and $k := 0$, we recover Von Karman's velocity field (1). Substitution of (5) into (8.2-2) yields the following system of differential equations defined on $[-d, d]$:

$$\begin{aligned}\nu h^{iv} + hh''' + gg' &= 0,\\ \nu g^{iv} + hg' - h'g &= 0,\\ \nu f''' + (hf')' - 1/2(h'f)' + 1/2(gk)' &= 0,\\ \nu k''' + (hk')' - 1/2(h'k)' + 1/2(gf)' &= 0.\end{aligned} \tag{8.10-6}$$

The requirement that the fluid adhere to the plates implies that

$$\begin{aligned}&h(-d) = h(d) = h'(-d) = h'(d) = 0,\\ &g(-d) = 2\Omega_{-d},\ g(d) = 2\Omega_d,\\ &f(-d) = f(d) = k(-d) = k(d) = 0.\end{aligned} \tag{8.10-7}$$

The structure of this system is most remarkable. The first two members of (6) and the first six of the boundary conditions (7) are precisely the conditions (2) and (3) that define the Von Kármán problem. Any solution g, h of that problem may be substituted into $(6)_{3,4}$, which then become linear differential equations for f and k. The order of the linear system is 6; their solutions are to satisfy the last four of the boundary conditions (7). Thus further conditions remain to be imposed if a unique solution is to be determined.

Whenever a solution of the axially symmetric problem exists, we can ask whether it is embedded in a broader class of solutions, generally not axially symmetric. Since the linear system for f and k is homogeneous and underdetemined, the answer is obviously yes. We need only consider the additional conditions

$$f'(-d) = k(-d) = 0. \tag{8.10-8}$$

If for given g and h the augmented system $(6)_{3,4}$ has a nontrivial solution f, k, then ℓf, ℓk is also a solution for every real number ℓ. If the augmented system does not have a nontrivial solution, then replacing (8) by

$$f'(-d) = \ell \neq 0, \quad k'(-d) = 0 \tag{8.10-9}$$

yields a unique solution of $(6)_{3,4}$. Therefore the axially symmetric solution is embedded in a one-parameter family of solutions of the full system of equations.

We next ask whether we can find a family of solutions for general Ω_{-d} and Ω_d that depend continuously on ν and ℓ and reduce when $\Omega_{-d} = \Omega_d$ to Berker's solution. Parter and Rajagopal[5] have proved that if g and h satisfy $(6)_{1,2}$ with given Ω_{-d} and Ω_d, then $(6)_{3,4}$ have solutions f, k that satisfy the boundary conditions $(7)_{7,8,9,10}$ as well as

$$f(0) = \ell, \quad k(0) = 0, \tag{8.10-10}$$

depend continuously on ν and ℓ, and reduce when $\Omega_{-d} = \Omega_d$ to Berker's solutions.

To discuss the nature of the solutions of these problems, it is best to introduce dimensionless statements corresponding to (6) and (7). Appropriate scaling parameters are Ω_{-d}/Ω_d and a Reynolds number defined as follows:

$$\mathcal{R} := \frac{d^2\Omega_d}{\nu}. \tag{8.10-11}$$

If $\zeta := z/d$ and

$$\begin{aligned} H(\zeta) : &= \sqrt{\mathcal{R}}\frac{h(z)}{\Omega_d d}, & G(\zeta) &:= \sqrt{\mathcal{R}}\frac{g(z)}{\Omega_d d}, \\ F(\zeta) : &= \sqrt{\mathcal{R}}\frac{f(z)}{\Omega_d d}, & K(\zeta) &:= \sqrt{\mathcal{R}}\frac{k(z)}{\Omega_d d}, \end{aligned} \tag{8.10-12}$$

then (6) becomes

$$\begin{aligned} \mathcal{R}^{-1/2}H^{\text{iv}} + HH''' + 4E^1GG' &= 0, \\ \mathcal{R}^{-1/2}G^{\text{iv}} + HG' - H'G &= 0, \\ \mathcal{R}^{-1/2}F''' + HF'' + (1/2)H'F' - (1/2)H''F + \mathcal{R}^{1/2}(GK)' &= 0, \\ \mathcal{R}^{-1/2}K''' + HK'' - (1/2)H''K + (1/2)H'K' - \mathcal{R}^{1/2}(GK)' &= 0, \end{aligned} \tag{8.10-13}$$

while (7) becomes

$$\begin{aligned} H(-1) = H(1) = H'(-1) = H'(1) = 0, \\ G(-1) = \Omega_{-d}/\Omega_d, G(1) = 1, \\ F(-1) = F(1) - K(-1) = K(1) = 0, \end{aligned} \tag{8.10-14}$$

and (8.10-11) becomes

$$F(0) = \alpha := \ell/(\Omega_d d), K(0) = 0. \tag{8.10-15}$$

For very small values of $\mathcal{R}$, Hastings[6] and Elcrat [7] have proved that $(10)_{1,2}$ has a unique axially symmetric solution subject to the boundary conditions $(11)_{1-6}$. For sufficiently large values of $\mathcal{R}$, Kreiss and Parter[8] have proved that the system

[5] S.V. Parter and K.R. Rajagopal, "Swirling flow between rotating plates," *Archive for Rational Mechanics and Analysis* 86 (1985): 305–15.

[6] S.P. Hastings, "On the existence theorems for some problems from boundary layer theory," *Archive for Rational Mechanics and Analysis* 38 (1970): 308–16.

[7] A.R. Elcrat, "On the swirling flow between rotating coaxial disks," *Journal of Differential Equations* 18 (1975): 423–30.

[8] H.O. Kreiss and S.V. Parter, "On the swirling flow between rotating coaxial disks," *Communications of Pure and Applied Mathematics* 36 (1983): 55–84.

$(10)_{1,2}$ subject to the conditions $(11)_{1-6}$ has several axially symmetric solutions. Parter and Rajagopal have proved that the system (10) subject to the conditions (11) has infinitely many solutions that are not axially symmetric. For small values of $\mathcal{R}$ the axially symmetric solution has a core that rotates at an angular speed equal to the average of the speeds of the two discs. On either side of the core there are narrow boundary layers. As $\mathcal{R}$ increases, the boundary layers become narrower. Numerical work suggests that as certain values of $\mathcal{R}$ there are as many as seven solutions, most of which exhibit no boundary layers.

Thus far we have restricted the discussion to flows induced by plates rotating about a common axis, but that is not necessary.

EXERCISE 8.10.1

If the axes of rotation are at $(a,0,d)$ and $(a,0,\text{-}d)$, the condition of adherence to the plates $z = \pm d$ replaces $(7)_{9,10}$ by $k(-d) = (1/2)\Omega_{-d}a$, $k(d) = 1/2\Omega_d a$. Show that

$$K(-1) = (1/2)\frac{\Omega_{-d}a}{\Omega_d d}, \quad K(1) = -(1/2)\frac{a}{d}. \tag{8.10-16}$$

Parter and Rajagopal find that each particular solution of the system $(10)_{1,2}$ gives rise to a one-parameter family of solutions of the system $(10)_{1-4}$. Therefore, if we are to discover the nature of the solutions of $(10)_{1-4}$, we must first determine the solutions of $(10)_{1,2}$. On that matter there is much conjecture and a bewildering array of numerical work.[9]

Some rigorous work has been done on the system $(10)_{1,2}$. For counter-rotating plates, namely, when $\Omega_d = -\Omega_{-d}$, McLeod and Parter[10] have proved the existence of a solution odd with respect to the plane $z = 0$. They have proved that G is monotone in the interval]-1, 1[and is exponentially small in a core region and that the shape of H in that interval is qualitatively like the negative of a sine function. They did not prove uniqueness, and later Kreiss and Parter showed that there are several other solutions when $\mathcal{R}$ is very large. Still less is known about the flow between disks rotating in the same sense. [11]

Steady flow between two infinite, rotating plates would seem to afford a rather simple problem, since it leads to no more than a solution of the system of four

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[9] The interested students may consult the survey articles by S.V. Parter, "On the swirling flow between co-axial disks: A survey" in *Theory and Applications of Singular Perturbation*, edited by W. Eckhaus and E. M. De Jager, Lecture Notes in Mathematics No. 942, (New York: Springer-Verlag, Berlin, Heidelberg, 1982), 258–80; K.R. Rajagopal, "Flow of viscoelastic fluids between rotating disks," *Theoretical and Computational Fluid Dynamics*, 3, (1992): 185–206.

[10] J.B. McLeod and S.V. Parter, "On the flow between two counter-rotating infinite plane disks," *Archive for Rational Mechanics and Analysis* 54 (1974): 301–27.

[11] J.B. McLeod and S.V. Parter, "The non-monotonicity of solutions in swirling flow," *Proceedings of the Royal Society of Edinburgh* 74 (1977): 161–82.

ordinary differential equations (10) subject to the conditions (11), but the foregoing pages show it to be far simple and far from settled.

8.11 Flow Driven by Coriolis Force

Ekman in his investigations of the influence of the earth's rotation upon ocean currents introduced the velocity field

$$\dot{x}_1 = u(x_3), \quad \dot{x}_2 = v(x_3), \quad \dot{x}_3 = 0. \tag{8.11-1}$$

To study the problem, the Cartesian coordinate system (x_1, x_2, x_3) should be regarded as having its origin on the free surface of the sea, with the x_3-axis directed vertically upward; the sea is taken as plane and infinitely deep.

The field (1) would be accelerationless in an inertial frame; here it is taken in a frame rotating with the earth. The components of the earth's angular velocity with respect to an inertial frame may be taken as ω_1, ω_2, ω_3, assumed constant. After using (2.3-10) to calculate the Coriolis acceleration, from (7.2-1) we get the following equations of motion:

$$\begin{aligned} 2\omega_3 v + \nu u'' &= \frac{1}{\rho}\partial_{x_1}\varphi, \\ -2\omega_3 u + \nu v'' &= \frac{1}{\rho}\partial_{x_2}\varphi, \\ 2(\omega_2 u - \omega_1 v) &= \frac{1}{\rho}\partial_{x_3}\varphi. \end{aligned} \tag{8.11-2}$$

From (2) we conclude that

$$\varphi = Ax_1 + Bx_2 + g(x_3), \tag{8.11-3}$$

in which A and B are arbitrary constants. This field is compatible with the assumption that φ depends only upon the depth. Here and henceforth we write x for x_3; then

$$\varphi = g(x), \tag{8.11-4}$$

and $(2)_{1,2}$ reduce to

$$\begin{aligned} 2\omega_3 v + \nu u'' &= 0, \\ -2\omega_3 u + \nu v'' &= 0. \end{aligned} \tag{8.11-5}$$

We write $f = u + iv$ and so express this system compactly as

$$f'' - 2i\frac{\omega_3}{\nu} f = 0. \tag{8.11-6}$$

Hence

$$f = Ce^{(1+i)\beta x} + De^{-(1+i)\beta x}, \quad \beta := \frac{\omega_3}{\nu}. \tag{8.11-7}$$

If a uniform tangential stress, representing for example the effect of the wind, acts upon the free surface, then

$$\mu u'(0) = T, \quad \mu v'(0) = 0. \tag{8.11-8}$$

If we suppose also that the fluid is at rest at $x = -\infty$, then

$$u \to 0 \text{ and } v \to 0 \text{ as } x \to -\infty. \tag{8.11-9}$$

The conditions (8) and (9) reduce (7) to

$$\begin{aligned} u &= \frac{T}{\sqrt{2}\mu\beta} e^{\beta x} \cos(\beta x - (1/4)\pi), \\ v &= \frac{T}{\sqrt{2}\mu\beta} e^{\beta x} \sin(\beta x - (1/4)\pi). \end{aligned} \tag{8.11-10}$$

Since $x < 0$, the velocity dies away exponentially from the surface, the flow being confined in the main to a boundary layer whose thickness may be taken as β^{-1}. The velocity at the surface subtends half a right angle upon the direction in which the flow is driven by the tangential stress T.

8.12 Serrin's Swirling Vortex

We consider a viscous fluid in motion about an infinite, straight vortex line normal to a plane boundary, to which the fluid adheres. The domain of flow is $r > 0$, $0 \le \theta < \frac{1}{2}\pi$, $0 \le \varphi < 2\pi$. We assume also that were the plane absent, the fluid would spin in an irrotational vortex centered upon the vortex line at angular speed inversely proportional to the square of the distance from it. Using a semiinverse method, we set up for study a class of flows described as follows in the physical components of a spherical coordinate system r, θ, ϕ:

$$\dot{x}_r = \frac{G(x)}{R}, \quad \dot{x}_\theta = \frac{F(x)}{R}, \quad \dot{x}_\phi = \frac{\Omega(x)}{R}, \tag{8.12-1}$$

in which $R := r \sin\theta$, the distance from the vortex line, and $x := \cos\theta$. Because (1) must be isochoric,

$$G = F' \sin\theta, \tag{8.12-2}$$

the prime denoting differentiation with respect to x. If $F = 0$ and $\Omega = C = \text{const.}$, the flow (1) reduces to the irrotational vortex of strength C.

Our problem is to choose F and Ω to make (1) a solution of Navier's dynamical equation and to interpret the resulting flow if such exists. We follow in outline a classic paper by Serrin.[12] Several of the early steps will be listed as exercises. Students who have difficulty in solving them should consult Serrin's paper, as they must do if they wish to follow the later parts of the analysis, only some outcomes of which are presented here.

EXERCISE 8.12.1

The spherical components of Navier's dynamical equation for this problem are

$$\begin{aligned} -FF'' - (F')^2 - (F^2 + \Omega^2)\operatorname{cosec}^2\theta &= -r^3\partial_r\varphi + \nu(F'''\sin^2\theta - 2F''\cos\theta), \\ -FF' - (F^2 + \Omega^2)\cot\theta\operatorname{cosec}\theta &= -r^2\sin\theta\,\partial_\theta\varphi - \nu F'''\sin^2\theta, \qquad (8.12\text{-}3) \\ -F\Omega' &= -r^2\partial_\phi\varphi + \nu\Omega''\sin^2\theta. \end{aligned}$$

From $(3)_3$ we see that $\partial_\phi\varphi$ is independent of ϕ. Hence φ is a linear function of ϕ, but since it must be periodic with period 2π in ϕ, it cannot depend upon ϕ at all. It follows from $(3)_1$ that there are functions of A and B such that

$$\varphi = \frac{A(x)}{r^2} + B(x). \qquad (8.12\text{-}4)$$

From $(3)_2$ we see that $B = \text{const}$. Thus (4) can be written as

$$\varphi = \frac{\pi(x)}{R^2} + \text{const}. \qquad (8.12\text{-}5)$$

and substitution into $(3)_1$ yields

$$-2\pi = F^2 + \Omega^2 + \left\{FF'' + F'^2 + \nu(F'''\sin^2\alpha - 2F''\cos\alpha)\right\}\sin^2\alpha. \quad (8.12\text{-}6)$$

EXERCISE 8.12.2 (LONG)

Eliminations of φ from (3) yield

$$\begin{aligned} \nu(1-x^2)F^{\mathrm{iv}} - 4\nu xF''' + FF''' + 3F'F'' &= -2\Omega\Omega'/(1-x^2), \\ \nu(1-x^2)\Omega'' + F\Omega' &= 0. \qquad (8.12\text{-}7) \end{aligned}$$

Students will have seen that these manipulations can be shortened, had we disposed of the spherical components of the condition of compatibility (8.2-2).

Our problem has been reduced to solution of the ordinary differential system (7) in the variable x, subject to appropriate boundary conditions. The condition of adherence $\dot{x} = 0$ must be satisfied when $\theta = \frac{1}{2}\pi$, and $\dot{x}_\phi$ must approach C/R

[12] J. Serrin, "The swirling vortex," *Philosophical Transactions of the Royal Society of London* 271A (1972): 325–60.

as $\theta \to 0$, C being the rotational momentum m per unit volume of the vortex described by $(1)_3$ alone. These conditions take the forms

$$\Omega = F = F' = 0 \text{ when } x = 0,$$
$$\Omega \to C \text{ as } x \to 1. \tag{8.12-8}$$

Also it is necessary to add the condition

$$F \to 0 \text{ as } x \to 1, \tag{8.12-9}$$

which stipulates that the vortex line $\theta = 0$ is neither a source nor a sink. Also the reduced pressure $\Pi(x) = \varphi R^2$ approaches the value $-\frac{1}{2}C^2$ as θ approaches zero, in keeping with the result for a line vortex.

EXERCISE 8.12.3
Let $\mathcal{S}$ be a surface of revolution obtained by revolving a meridian curve $\mathcal{C}$ about the vortex line $\theta = 0$. If $\psi := rF(x)$, then show that

$$\dot{x}_r = -\frac{1}{r^2 \sin\theta}\partial_\theta \psi, \ \dot{x}_\theta = \frac{1}{r \sin\theta}\partial_r \psi, \tag{8.12-10}$$

and the flux through $\mathcal{S}$ is $2\pi|\psi_2 - \psi_1|$, in which ψ_1 and ψ_2 are the values of ψ at the endpoints of the curve $\mathcal{C}$. If $\mathcal{C}$ joins two distinct points on the vortex line, then $\psi_1 = \psi_2$, so (9) follows. The streamlines lie on the surfaces $\psi = \text{const}$.

We have reduced our problem on the sixth-order differential system (7) subject to the five conditions (8) and (9). The constants ν and C are assigned. We expect to find a one-parameter family of solutions. We shall now reduce the problem to a statement in terms of integro-differential equations.

First, integrating $(7)_1$ three times yields

$$2\nu(1 - x^2)F' + 4\nu x F + F^2 = -\int_0^x dy \int_0^y dz \int_0^z \frac{4\Omega\Omega'}{1 - w^2} dw + Px^2 + Qx; \tag{8.12-11}$$

P and Q are constants of integration, and a third one has been annulled by use of $(8)_{2,3}$.

EXERCISE 8.12.4
Show that the triple integral in (11) can be rewritten as

$$2\int_0^x \frac{(x-t)^2}{1-t^2}\Omega\Omega' dt; \tag{8.12-12}$$

integration by parts and using the boundary condition $\Omega(0) = 0$ yields

$$2\int_0^x \frac{(x-t)(1-xt)}{(1-t^2)^2}\Omega^2 dt. \tag{8.12-13}$$

Since Ω is bounded on [0,1], this integral converges as $x \to 1$ to

$$2\int_0^1 \frac{\Omega^2}{(1+t)^2}dt. \tag{8.12-14}$$

By use of (7) and (9) we find that

$$(1-x)F' \to \text{const. as } x \to 1; \tag{8.12-15}$$

use of (9) again shows that the constant is naught. Thus (11) yields

$$P + Q = 2\int_0^1 \frac{\Omega^2}{(1+t)^2}dt. \tag{8.12-16}$$

Following Serrin, we first choose P as a basic parameter and use (16) to eliminate Q from (11), so we obtain

$$2\nu(1-x^2)F' + 4\nu x F + F^2 = G,$$
$$G(x) := 2(1-x)^2\int_0^x \frac{t\Omega^2}{(1-t^2)^2}dt + 2x\int_x^1 \frac{\Omega^2}{(1+t)^2} - (x-x^2)P. \tag{8.12-17}$$

With the change of variables $F := 2\nu(1-x^2)f$, we obtain the final system:

$$f'^2 + f^2 = \tfrac{C^2}{4\nu^2(1-x^2)^2}G,$$
$$\Omega'' + 2f\Omega' = 0 \tag{8.12-18}$$

in [0,1], and the additional conditions are

$$f = \varpi = 0 \text{ when } x = 0, \Omega \to C \text{ as } x \to 1. \tag{8.12-19}$$

We notice that both the angular momentum C and the viscosity ν have dimension of L^2/T, and thus a natural choice for the Reynolds number of these flows is

$$\mathcal{R} := \frac{|C|}{2\nu}. \tag{8.12-20}$$

Serrin shows by means of a change of variable that there is no loss in generality in setting $C = 1$. He also provides some motivation for generalizing the analysis to turbulent flow wherein ν is to be interpreted as the kinematic eddy viscosity.

EXERCISE 8.12.5

The rotational momentum Ω increases monotonically from 0 to C as x increases from 0 to 1. The statement of this exercise is drawn directly from $(18)_2$ and the fact that Ω' is always of one sign.

THEOREM 8.1

Let f and ϖ satisfy (18) and the additional condition (19). Then F is a solution of (7) subject to the additional conditions (8) and (9); moreover,

$$\pi(0) = -\frac{1}{2}PC^2, \pi(1) = -\frac{1}{2}C^2. \tag{8.12-21}$$

Also

$$\varphi \to -\frac{PC^2}{2R^2} + \text{const as } \theta = 0,$$
$$\varphi = -\frac{C^2}{2R^2} + \text{const. when } \theta = \frac{1}{2}\pi. \tag{8.12-22}$$

This conclusion shows, among other things, that the solution of (18) subject to (19) leads to a solution of (7) subject to (8) and (9).

Before getting into a discussion of the existence of solutions of the above equations, we shall discuss some of their qualitative properties.

EXERCISE 8.12.6

Show that the function $G(x)$ satisfies

$$G(0) = 0, G'(0) = 2\int_0^1 \frac{\Omega^2 dt}{(1+t^2)} - P, \tag{8.12-23}$$
$$G(1) = 0, G'(1) = P - 1, \tag{8.12-24}$$

and $G''(x) < 0$ for $0 < x < 1$.

EXERCISE 8.12.7

Show that the function $G(x)$ is such that

$$G'(0)(x - x^2) < G(x) < (1 - P)(x - x^2), 0 < x < 1. \tag{8.12-25}$$

As a consequence, we conclude that for $0 < x < 1$

$$G(x) > 0 \text{ when } G'(0) \geq 0,$$
$$G(x) < 0 \text{ when } P \geq 1, \tag{8.12-26}$$

and when $G'(0) < 0$ and $P < 1$, G is first negative, has a single zero in the interval $0 < x < 1$, and is positive thereafter. We thus conclude that G is either concave, convex, or initially convex and then concave.

EXERCISE 8.12.8

If $P < 1$, then $f \le \frac{1}{4}(1-P)^2 \log\left(\frac{1}{1-x}\right)$.

Also if $Q \ge 0$, then f is positive, whilst if $Q < 0$, then f is either everywhere negative or else first negative and positive thereafter. Finally, verify that if $P \ge 1$, then f is negative and decreasing.

With the aid of Exercises 6, 7, and 8 we can establish the following theorems that describe the behavior of the function f.

THEOREM 8.2

Let f and Ω make up a solution of (18) subject to (19).

(i) *Suppose that $P < 1$. If $G'(0) \ge 0$, then $f > 0$ and Ω increases and is concave. If $G'(0) < 0$, then f is first negative, then has a single zero in [0,1], and is positive thereafter.*

   Correspondingly, Ω is first convex and then concave.

(ii) *Suppose that $P \ge 1$. Then f is negative and decreases, while ϖ increases and is convex.*

(iii) *When $P = 1$, f tends to a finite limit as $x \to 0$.*

In other cases,

$$f \sim \frac{1}{4}(1-P)\mathcal{R}^2 \log \frac{1}{1-x}. \tag{8.12-28}$$

Serrin illustrates the streamlines of these cases for particular values of P. In some instances fluid flows down near the vortex line and along the plate until the two streams meet and are turned upward to form a tight funnel pointing straight ahead. In another instance the fluid sweeps in along the plate and is turned upward to form a vertical spout.

Serrin's conclusions can be summarized as follows:

Let f be a solution of (18) subject to (10). Then f is either everywhere positive, everywhere negative, or first negative and then positive.

A. *For positive solutions the radial velocity is outward near the plane and downward near the vortex line.*

B. *For solutions at first negative and then positive, the radial velocity is inward near the plane and downward near the vortex line. This general motion toward the origin is balanced by a compensating outflow near the streamcone of angle $\cos^{-1} a$, a being the zero of f.*

C. *For negative solutions the radial velocity is inward near the plane and upward near the vortex line.* Later Serrin proves that such solutions exist only for small values of $\mathcal{R}$. For positive f, solutions of (18) subject to (19) have a boundary layer in which as $\mathcal{R}$ becomes large the function Ω uniformly approaches 1 on compact subsets of $]0,1]$. The component $\dot{x}_\varphi$ is then arbitrarily near that of an irrotational vortex except for a thin layer near the plane. The radial component of velocity is quite large.

Serrin shows that there are ranges of parameters for which no solution of (18) subject to (19) exists.

1. If $P \geq 1$ and $P\mathcal{R}^2 > \frac{g}{2}\mu_1^2$, where μ_1 is the smallest zero of the Bessel function $J_{\frac{-1}{3}}$, then the system (18) subject to (19) has no solution.

2. Let a be a fixed number in $]0,1[$. There is no solution f such that $f(a) = 0$ unless

$$P \leq \frac{1}{a}\left\{1 + \frac{1-a}{a^2}\log(1-a^2)\right\}. \tag{8.12-29}$$

3. There is no positive solution unless

$$P \leq K(\mathcal{R}), \tag{8.12-30}$$

where K is a positive function that increases monotonically from $3 - 4\log 2$ to 1 as $\mathcal{R}$ increases from 0 to ∞.

The existence theory for (18) subject to (19) is difficult. Let the parameters $\mathcal{R}$ and P be fixed at the values $\tilde{R}$ and $\tilde{P}$, and let $\tilde{f}$ and Ω be such that $\tilde{f}(0) = \tilde{\Omega}(0) = 0$ and

$$\tilde{f} = O\left(\log\frac{1}{1-x}\right), \qquad 0 \leq \tilde{\Omega} \leq 1. \tag{8.12-31}$$

Then $\tilde{f}$ and $\tilde{\Omega}$ will be called a *subsolution* of (18) subject to (19) if on $[0,1[$

$$\begin{aligned} \tilde{f}' + \tilde{f}^2 &\leq \tilde{R}^2 \tfrac{\tilde{G}(x)}{(1-x^2)^2}, \\ \tilde{\Omega}'' + 2\tilde{f}\tilde{\Omega} &\geq 0, \end{aligned} \tag{8.12-32}$$

in which $\tilde{G}$ is the function G given by $(17)_2$ with Ω and P given by $\tilde{\Omega}$ and $\tilde{P}$, respectively.

THEOREM 8.3

For given $\tilde{R}$ and $\tilde{P}$, let $\tilde{f}$ and $\tilde{\Omega}$ be a subsolution. Then there is a solution f and Ω of (18) subject to (19) for the same $\mathcal{R}$ and all values of P not exceeding $\tilde{P}$. Moreover,

$$f \geq \tilde{f}, \Omega \geq \tilde{\Omega}. \tag{8.12-33}$$

The proof employs an intricate procedure of successive approximation.

Some corollaries deliver ranges of the parameters for which solutions do exist and others for which they do not. For example, if $P \leq 3 - 4\log 2$, there are positive solutions. Also there is a constant λ, pretty near 2.85, such that the system is solvable if $P\mathcal{R}^2 < \lambda^2$ but is not solvable if $(P-1)\mathcal{R}^2 > \lambda$. If $P < 3 - 4\log 2$, there is a solution with positive f, while if $P \geq 1$ and $P\mathcal{R}^2 < 8.2$, there is a decreasing solution with negative f. The type of solution in which f is at first negative, then has a single zero, and thereafter is positive, remains to be established, as follows.

From a step in his proof of Theorem 2, Serrin remarks that solutions of the kind he now seeks must satisfy the condition

$$G'(0) = 2\int_0^1 \frac{\Omega^2}{(1+t)^2}dt - P < 0. \tag{8.12-34}$$

He next writes Q for $G'(0)$ and takes Q as the basic parameter. Thus

$$P = 2\int_0^1 \frac{\Omega^2}{(1+t)^2}dt - Q. \tag{8.12-35}$$

The system to be solved is (18) subject to the conditions (29) and (30). Eliminating P from (29) by use of (30), we obtain

$$G(x) = -2\int_0^x \frac{(x-t)(1-tx)}{(1-t^2)^2}\Omega^2 dt + 2x^2\int_0^1 \frac{\Omega^2}{(1+t)^2}dt + (x-x^2)Q. \tag{8.12-36}$$

By using a fixed-point theorem, Serrin proves the following theorem.

THEOREM 8.4

The system (18) subject to the conditions (19) is solvable, with the solution given by (31), provided that

$$Q\mathcal{R}^2 > \lambda^2 \tag{8.12-37}$$

(the constant λ was introduced in the first corollary following Theorem 3).

A corollary states that for each $\mathcal{R}$ there are solutions f of (18) subject to (19) such that f is first negative, then has a zero in]0,1[, and is positive thereafter.

This theorem guarantees that there is a solution provided $G'(0) < 0$, which corresponds to $P < 1$. These solutions are positive. Serrin goes on to prove that, given a fixed $\mathcal{R}$, there is a positive solution provided;

$$P \geq 1 - 12(\mathcal{R})^{-1/2}[1 + |\log\mathcal{R}|]. \tag{8.12-38}$$

On the basis of numerical calculations, Serrin provides a plot $(\mathcal{R})^{-2}$ versus P in which he delineates the domains in which the different types of solutions are possible. It is seen that if P is sufficiently small and $\mathcal{R}$ is sufficiently large, then we always have a positive solution. On the other hand, if both P and $\mathcal{R}$ are sufficiently large, no solution is possible. In fact, for any fixed value of $\mathcal{R}$, if P is sufficiently large, there will be no solution. For intermediate values of $\mathcal{R}$ and P, solutions that are initially negative and then positive and solutions that are everywhere negative are possible.

Consideration of uniqueness is notably absent in Serrin's study of the swirling vortex. The paper concludes with diagrams obtained by numerical calculation and with photographs of waterspouts and tornados, to which the analysis may apply.

Theorems of general existence and uniqueness of solutions for boundary-value and initial-value problems apart, in the theory of the Navier-Stokes equations all problems solved precisely so far rest upon particular reductions to systems of ordinary differential equations. As in Serrin's treatment of the swirling vortex, these differential equations are usually nonlinear and very hard to solve.

8.13 Stability and Uniqueness

Recent years have seen great strides in the study of hydrodynamic stability based upon the Navier-Stokes theory.[13] In former times nearly all of the effort was put into analysis of linearized theories, which provide sufficient conditions for instability.[14] Bifurcation theory, in contrast, obtains properties of all solutions that can arise from the instability of a particular, given solution. Between these extremes lies the rigorous theory of stability in norm. We shall now consider some examples.

We consider first a body of Navier-Stokes fluid undergoing a given, basic flow in a fixed, bounded, closed domain $\mathcal{V}$ in space and subjected to arbitrary disturbance. In general, fluid will enter and leave $\mathcal{V}$ through $\partial\mathcal{V}$. The shape of $\mathcal{V}$ may change in time, but its volume V will remain constant. The arguments concern certain integral norms of the squared speed of a disturbance; such a norm is often called an energy E, and stability defined in terms of it is called stability in energy.

First we state and prove three classic theorems of Serrin, who based his analysis in part on the pioneering work of Reynolds and Orr as well as the early rigorous studies by E. Hopf and T.Y. Thomas. In following the proofs of the first theorems, the student will notice similarities to the work presented Section 7.6. The procedures there were influenced by the prior studies of Serrin, some of which we now set forth.

We write $\dot{\mathbf{x}}$ for the basic flow, to which φ is associated as usual, and we suppose that these satisfy Navier's dynamical equation (8.2-1). To prove Serrin's theorems we presume $\dot{\mathbf{x}}$ and φ smooth in the following senses:

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[13]The survey by D.D. Joseph, *Stability of Fluid Motions*, Springer Tracts in Natural Philosophy, vol. 27 and 28, (Berlin, Heidelberg, and New York: Springer-Verlag, 1976), presents and interrelates developments in all kinds of hydrodynamic stability by him and others through 1975.

[14]C.C. Lin, *The Theory of Hydrodynamic Stability*, (Cambridge: Cambridge University Press, 1955).

(i) $\dot{\mathbf{x}}$ and its first derivatives are continuous in $\mathcal{V} \times [0, T]$, $T > 0$, and $\dot{\mathbf{x}}$ has continuous spatial second derivatives in $\mathcal{V}$.

(ii) φ and grad φ are continuous in $\mathcal{V} \times [0, T]$.

We suppose that $\dot{\mathbf{x}}^*$ and φ^*, perhaps different from $\dot{\mathbf{x}}$ and φ, in $\nu \times [0, T]$ satisfy Navier's equations (8.2-1) with the same value of ν and that $\dot{\mathbf{x}}^*$ and φ^* enjoy the smoothness specified earlier for $\dot{\mathbf{x}}$ and $\dot{\varphi}$. The initial conditions for $\dot{\mathbf{x}}^*$ and φ^* will generally be different from those prescribed for $\dot{\mathbf{x}}$ and φ. We call the differences the perturbances:

$$\delta\dot{\mathbf{x}} := \dot{\mathbf{x}}^* - \dot{\mathbf{x}}, \delta\varphi := \varphi^* - \varphi. \tag{8.13-1}$$

Hence, if $t \in [0, T]$,

$$\delta\dot{\mathbf{x}} = 0 \text{ on } \partial\mathcal{V}, \tag{8.13-2}$$

while if $t = 0$, then

$$\delta\dot{\mathbf{x}} = \delta\dot{\mathbf{x}}_0, \tag{8.13-3}$$

a prescribed function of $\mathbf{x}$ in $\mathcal{V}$.

We use the asterisk and the δ to denote also the values of linear operations on $\dot{\mathbf{x}}^*$ and $\delta\dot{\mathbf{x}}$. Thus the velocity gradients, stretching, and spins of $\dot{\mathbf{x}}^*$ and $\delta\dot{\mathbf{x}}$ are $\mathbf{G}^*$, $\mathbf{D}^*$, $\mathbf{W}^*$ and $\delta\mathbf{G}$, $\delta\mathbf{D}$, $\delta\mathbf{W}$, respectively.

Subtracting the Navier-Stokes equation satisfied by $\dot{\mathbf{x}}$ and φ from that for $\dot{\mathbf{x}}^*$ and φ^* yields

$$\delta\dot{\mathbf{x}} + \mathbf{G}\delta\dot{\mathbf{x}} + (\delta\mathbf{G})\dot{\mathbf{x}} = -\operatorname{grad}\delta\varphi + 2\nu \operatorname{div}\delta\mathbf{W}. \tag{8.13-4}$$

The prime denotes ∂_t in the spatial description. The scalar product of this equation and $\delta\dot{\mathbf{x}}$ is

$$\begin{aligned} &\frac{1}{2}(|\delta\dot{\mathbf{x}}|^2)' + \delta\dot{\mathbf{x}} \cdot \mathbf{G}\delta\dot{\mathbf{x}} + (\delta\dot{\mathbf{x}}) \cdot (\delta\mathbf{G})\dot{\mathbf{x}}^* \\ &\quad = -\delta\dot{\mathbf{x}} \operatorname{grad} \delta\varphi + 2\nu(\delta\dot{\mathbf{x}}) \cdot \operatorname{div}\delta\mathbf{W}. \end{aligned} \tag{8.13-5}$$

Recalling that $\dot{\mathbf{x}}$ and $\dot{\mathbf{x}}^*$ are both solenoidal, we can rewrite (5) as

$$\begin{aligned} \frac{1}{2}(|\delta\dot{\mathbf{x}}|^2)' &= -\delta\dot{\mathbf{x}} \cdot \mathbf{D}\delta\dot{\mathbf{x}} - \nu|\delta\mathbf{G}|^2 + \operatorname{div}\mathbf{z}, \\ \mathbf{z} &:= \delta\varphi\delta\dot{\mathbf{x}} + \frac{1}{2}|\delta\dot{\mathbf{x}}|^2\mathbf{x}^* - \delta\mathbf{G}^T\delta\dot{\mathbf{x}}. \end{aligned} \tag{8.13-6}$$

Noting that $\mathbf{z} = \mathbf{0}$ on $\partial\mathcal{V}$, we integrate (6) over $\mathcal{V}$, use the divergence theorem, and so obtain Serrin's *basic balance*, which, he stated, "may be traced" to the work of Reynolds and Orr:

$$\dot{E} = -\int_{\mathcal{V}} (\delta\dot{\mathbf{x}} \cdot \mathbf{D}\delta\dot{\mathbf{x}} + \nu|\delta\mathbf{G}|^2)dV, \; E := \frac{1}{2}\int_{\mathcal{V}} |\delta\dot{\mathbf{x}}|^2 dV. \tag{8.13-7}$$

The foregoing derivation of (7) presumes $\mathcal{V}$ bounded. That balance holds also for unbounded domains if the perturbance is assumed spatially periodic at each instant. For unbounded domains it holds also if $\dot{\mathbf{x}}$, $\dot{\mathbf{x}}^*$, φ, and φ^* are presumed to satisfy appropriate conditions of decay as $\mathbf{x} \to \infty$. Theorem 5 delivers uniqueness under rather weak assumptions.

The first term in (7) is independent of viscosity; it delivers the change of norm due to interaction of the perturbance and the stretching of the basic flow, the same as for an Eulerian fluid. The second term, proportional to ν, delivers the change due to the difference of the gradients of the two flows, moderated by the viscosity.

Since the flow is divergence free and $\dot{\mathbf{x}} = \mathbf{0}$ on the boundary we can replace $|\delta\mathbf{G}|^2$ by $2|\delta\mathbf{D}|^2$ in $(7)_1$. Thus we can say that all change of E is governed by the interior stretching of the two flows. Alternatively, we may replace $|\delta\mathbf{G}|^2$ by $2|\delta\mathbf{W}|^2$, and the statement produced by solving the next exercise allows us also to replace the integrand $\delta\dot{\mathbf{x}} \cdot \mathbf{D}\delta\dot{\mathbf{x}}$ by one proportional to $\delta\mathbf{W}$.

EXERCISE 8.13.1

$$\int_{\mathcal{V}} \delta\dot{\mathbf{x}} \cdot \mathbf{D}\delta\dot{\mathbf{x}} dV = -2\int_{\mathcal{V}} \dot{\mathbf{x}} \cdot \delta\mathbf{W}\delta\dot{\mathbf{x}} dV. \tag{8.13-8}$$

Thus we may say equally well that all change of E is governed by the difference of the interior spins of the two flows and again that change is the sum of the two: one independent of viscosity and one expressing the effects of viscosity.

While up to now in this section we have considered only solutions of Navier's dynamical equation, we here state and prove a lemma concerning a broader class of continuously differentiable, time-dependent vector fields $\mathbf{v}$ such that $\operatorname{div} \mathbf{v} = \mathbf{0}$ in a region $\mathcal{V}$ and $\mathbf{v} = \mathbf{0}$ on $\partial\mathcal{V}$. Of course $\delta\dot{\mathbf{x}}$ is an instance of $\mathbf{v}$, so our conclusions will apply to it *a fortiori*. We presume that $\mathbf{v}$ satisfies the requirements laid down for $\dot{\mathbf{x}}$.

LEMMA 8.5

Let $\mathbf{D}$ be a field of symmetric tensors, and let m be a lower bound of the proper numbers of $\mathbf{D}$ in $\mathcal{V} \times [0, T]$. Let $\mathbf{v}$ be a vector field, let C be an upper bound for the Poincaré coefficients of $\mathcal{V}$ in [0,T], and let k be any positive constant. Then

$$-\int_{\mathcal{V}} (\mathbf{v} \cdot \mathbf{D}\mathbf{v} + k|\nabla\mathbf{v}|^2) dV \leq \left(m - \frac{k}{C}\right) \int_{\mathcal{V}} |\mathbf{v}|^2 dV. \tag{8.13-9}$$

PROOF At each point in $\mathcal{V}$ and at each time, $\mathbf{u} \cdot \mathbf{D}\mathbf{u} \geq -cu^2$, $c = const. \geq 0$, $\mathbf{u}$ arbitrary. Thus for m and $\mathbf{v}$ satisfying the hypothesis of the Lemma throughout

[0,T]

$$\int_{\mathcal{V}} \mathbf{v} \cdot \mathbf{D}\mathbf{v}\, dV \geq -m \int_{\mathcal{V}} |\mathbf{v}|^2 dV. \tag{8.13-10}$$

At each instant

$$\int_{\mathcal{V}} |\mathbf{v}|^2 \leq C_p \int_{\mathcal{V}} |\nabla\mathbf{v}|^2 dV. \tag{7.6-7}$$

Consequently, for the interval of time $[0,T]$ there is a positive constant C such that

$$\int_{\mathcal{V}} \mathbf{v} \cdot \mathbf{D}\mathbf{v} \geq C \int_{\mathcal{V}} |\nabla\mathbf{v}|^2 dV. \tag{8.13-11}$$

From the negative of (10) we subtract k/C times (11) and so obtain (9).

LEMMA 8.6

If $t \in [0, T]$, then

$$E(t) \leq E(0)e^{2(m-\nu/C)t}. \tag{8.13-12}$$

PROOF In (9) we may take $\delta\dot{\mathbf{x}}$ for $\mathbf{u}$ and ν for k. Recalling $(7)_1$, we obtain

$$\dot{E} \leq aE, a := 2\left(m - \frac{\nu}{E}\right). \tag{8.13-13}$$

If $t \in [0, T]$, from $(13)_1$ we see that

$$\begin{aligned} 0 &\geq \int_0^t (\dot{E} - aE)e^{-aw} dw, \\ &= \int_0^t \frac{d}{dw}(Ee^{-aw}) dw, \\ &= E(t)e^{-at} - E(0). \end{aligned} \tag{8.13-14}$$

THEOREM 8.7 (Foa, Serrin)

Uniqueness of the initial-value problem. If the initial perturbance, $\delta\dot{\mathbf{x}} = \mathbf{0}$, then $E(0) = 0$, and for all t in $[0,T]$ it follows that $E(t) = 0$. Hence $\delta\dot{\mathbf{x}} = \mathbf{0}$ at each t in $[0,T]$.

The proof is immediate because we have assumed that $E(0) = 0$ in (12) and (14). It makes no use of $(13)_2$. Thus Theorem 1 applies to all incompressible Navier-Stokes fluids regardless of ν.

THEOREM 8.8 Decay (Serrin). *If $t \in [0,T]$,*

$$m < \frac{\nu}{C}, \tag{8.13-15}$$

then $E(t)$ decreases exponentially. If $T = \infty$, then

$$E(t) \to 0 \text{ monotonically as } t \to \infty. \tag{8.13-16}$$

PROOF Immediate from (12).

When ν and $\mathcal{V}$ are assigned, (15) provides a criterion of distortion: If the fluid undergoing the basic flow does not at any time contract too rapidly in some direction, it is stable in energy. Serrin expresses (15) in terms of a Reynolds number (8.4-2), taking V as the maximum speed of the basic flow and L as the maximum diameter of $\mathcal{V}$. By calculation he found the criterion $\mathcal{R} < 5.71$ sufficient for universal stability. For particular classes of flows, similar and sharper bounds had been established before Serrin's, notably by Leray, Kampe de Feriet, and Berker.

THEOREM 8.9 Uniqueness of steady flows (Serrin). *If (15) holds, at most one steady flow satisfies Navier's equation and the boundary conditions (2).*

PROOF $E(t) = E(0)$. If (15) holds, from (12) we see that $E(t) < E(0)$, which is not possible. Hence $E(0) = 0$.

The proofs of these three theorems rest heavily upon Lemma 1, which follows from vector analysis and tensor analysis alone, unaffected by the Navier-Stokes theory. Consequently conditions sufficient for uniqueness or decay established by use of (9) or other bounds similarly obtained are unnecessarily weak. They allow for an enormous class of disturbances that cannot exist. Serrin's discovery provoked much effort, at the cost of much mathematical difficulty, to obtain a condition weaker than (15), and Joseph obtained such a condition.[15] This method has the advantage of leaving the proofs essentially unchanged while sharpening conclusions through replacement of (15) by some weaker condition. Later research has delivered still stronger estimates.

Serrin's theorems estimate the behavior of E, a kind of mean. Much of Hopf's pioneering effort was directed toward proof that stability in norm required the disturbance to vanish. This difficult matter has been cleared by Rosso.[16]

We turn now to solutions of Navier's dynamical equation that describe what has occurred in the past, before the start of the forward flow. It was long known that not all backward solutions depend continuously upon their defining initial data but

[15] D.D. Joseph, *Stability of Fluid Motions,* 14–15.

[16] F. Rosso, "Variational methods for pointwise stability of viscous fluid motions," *Archive for Rational Mechanics and Analysis* 86 (1984): 181–95, and "Pointwise unconditional attractivity of solutions of the Navier-Stokes equations with respect to perturbations of Hopf type," *Revue Roumain des Mathématiques Pures et Appliquées* 31 (1986): 51–64.

that such solutions belonging to some particular classes do. Knops and Payne[17] constructed one such class. In it the speed, spin, and acceleration of the basic flow and the speed of the perturbed flow are uniformly bounded in the domain of their definition, which is $\mathcal{V} \times [-T, 0]$, T being a positive constant. Neither the basic flow $(\dot{\mathbf{x}}, \varphi)$ nor the perturbed flow $(\dot{\mathbf{x}}^*, \varphi^*)$ is assumed to exist, but if they do, they are assumed to satisfy identical initial conditions and the boundary conditions (2) and (3). The student will recall the definition of the perturbance $\delta\dot{\mathbf{x}}$ and its initial value $\delta\dot{\mathbf{x}}$ and its initial value $\delta\dot{\mathbf{x}}_0$ in $\mathcal{V}$. Knops and Payne's bounds are defined as follows in terms of supremum over $\mathcal{V} \times [-T, 0]$ and of arbitrary constants N and M:

$$\begin{aligned} sup(|\dot{\mathbf{x}}|^2 + |\mathbf{W}|^2|\dot{\mathbf{x}}'|^2) &\leq N^2, \\ sup|\dot{\mathbf{x}}^*|^2 &\leq M^2. \end{aligned} \tag{8.13-17}$$

THEOREM 8.10 (Continuous dependence on data for solutions backward in time)

Let the stated conditions be satisfied for all t in [0,T]. Then there are a positive constant K and a function λ of t with values in [0,1] such that

$$2E \leq K\left(\int_{\mathcal{V}} |\delta\dot{\mathbf{x}}_0|^2 dV\right)^{\lambda}. \tag{8.13-18}$$

PROOF From $(7)_2$ we see that

$$\begin{aligned} \dot{E} &= \int_{\mathcal{V}} (\delta\dot{\mathbf{x}}_o) \cdot (\delta\dot{\mathbf{x}})' dV, \\ \ddot{E} &= \int_{\mathcal{V}} (\delta\dot{\mathbf{x}}' \cdot \delta\dot{\mathbf{x}}' + \delta\dot{\mathbf{x}} \cdot \delta\dot{\mathbf{x}}'')dV, \\ &= \int_{\mathcal{V}} (\delta\dot{\mathbf{x}}' \cdot \delta\dot{\mathbf{x}}')dV + \int_{\mathcal{V}} \delta\dot{\mathbf{x}}[\nu\Delta\delta\dot{\mathbf{x}} - \text{grad } \delta\varphi \\ &\quad - (\delta\mathbf{G})\dot{\mathbf{x}}^* - \mathbf{G}\delta\dot{\mathbf{x}}]' dV; \end{aligned} \tag{8.13-19}$$

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[17]R.J. Knops and L.E. Payne, "On the stability of solutions of the Navier-Stokes equations backward in time," *Archive for Rational Mechanics and Analysis* 29 (1968): 331–35. A treatment starting from a different set of conditions is given by G.P. Galdi and B. Straughan, "Stability of solutions to the Navier-Stokes equations backward in time," *Archive for Rational Mechanics and Analysis* 101 (1988): 107–14. This paper includes a brief history of the problem.

the last step follows by use of (4). From the boundary conditions and use of the divergence theorem we obtain

$$\ddot{E} = 2\int_{\mathcal{V}} |\delta\dot{\mathbf{x}}'|^2 dV + 2\int_{\mathcal{V}} [(\delta\mathbf{G})\dot{\mathbf{x}}^* \cdot \delta\dot{\mathbf{x}}']\, dV \tag{8.13-20}$$

$$+ \int_{\mathcal{V}} (\mathbf{W}\delta\dot{\mathbf{x}}) \cdot \delta\dot{\mathbf{x}}'\, dV + \int_{\mathcal{V}} [(\delta\mathbf{G})\delta\dot{\mathbf{x}}] \cdot \dot{\mathbf{x}}\, dV. \tag{8.13-21}$$

Next, by virtue of the boundary conditions and the divergence theorem, we see that

$$\int_{\mathcal{V}} \delta\dot{\mathbf{x}} \cdot \left[(\delta\mathbf{G})\dot{\mathbf{x}}^* + \frac{1}{2}\mathbf{W}\delta\dot{\mathbf{x}}\right] dV = 0, \tag{8.13-22}$$

so $(19)_1$ can be expressed as follows:

$$\dot{E} = \int_{\mathcal{V}} \delta\dot{\mathbf{x}} \cdot (\delta\dot{\mathbf{x}})' dV + \frac{1}{2}\int_{\mathcal{V}} \delta\dot{\mathbf{x}} \cdot \left[(\delta\mathbf{G})\dot{\mathbf{x}}^* + \frac{1}{2}\mathbf{W}\delta\dot{\mathbf{x}}\right] dV = 0. \tag{8.13-23}$$

To help students understand the following analysis, we discuss briefly the essential idea underlying the proof entering into the details. The argument rests upon logarithmic convexity. A differential inequality that can be integrated leads to a bound on an appropriate Liapunov function, which in the present instance is E. For the moment, let us suppose E does not vanish at any t. Thus, if we can show that there are positive constants K_1 and K_2 such that

$$\ddot{E} - \frac{\dot{E}^2}{E^2} \geq K_1\frac{\dot{E}}{E} - K_2, \tag{8.13-24}$$

rearrangement yields

$$\left(e^{-K_1 t}\frac{\dot{E}}{E}\right) \leq K_2 e^{-K_1 t}. \tag{8.13-25}$$

With the change of variable $\sigma = e^{K_2 t}$, (24) becomes

$$\frac{d}{d\sigma}\left(\frac{1}{E}\frac{dE}{d\sigma}\right) \geq -K_2(K_1\sigma)^{-2}. \tag{8.13-26}$$

This inequality can be integrated twice to obtain a bound for E in terms of the initial perturbance $\delta\mathbf{x}_0$. First we establish the differential inequality (25).

EXERCISE 8.13.2 (Knops and Payne)
The relations $(7)_2$, $(19)_3$, and (22) imply that

$$E\ddot{E} - \dot{E}^2 = \left\{\int_{\mathcal{V}} |\delta\dot{\mathbf{x}}|^2 dV \int_{\mathcal{V}} |\delta\dot{\mathbf{x}}' + \frac{1}{2}(\delta\mathbf{G})\dot{\mathbf{x}}^* + \frac{1}{4}\mathbf{W}\delta\dot{\mathbf{x}}|^2 dV\right.$$

$$- \left(\int_{\mathcal{V}} \delta \dot{\mathbf{x}} \cdot \left[\delta \dot{\mathbf{x}}' + \frac{1}{2}(\delta \mathbf{G})\dot{\mathbf{x}}^* + \frac{1}{4}\mathbf{W}\delta\dot{\mathbf{x}} \right] dV \right)^2 \Bigg\}$$

$$- \frac{1}{4} \int_{\mathcal{V}} |\delta \dot{\mathbf{x}}|^2 dV \int_{\mathcal{V}} |(\delta \mathbf{G})\dot{\mathbf{x}}^* + \frac{1}{2}\mathbf{W}\delta\dot{\mathbf{x}}|^2 dV + \frac{1}{2} \int_{\mathcal{V}} |\delta \dot{\mathbf{x}}|^2 dV \int_{\mathcal{V}} [(\delta \mathbf{G})\delta\dot{\mathbf{x}}] \cdot \dot{\mathbf{x}}' dV. \tag{8.13-27}$$

Taking the scalar product of (4) and $\delta\dot{\mathbf{x}}$ and using the initial conditions and boundary conditions (3) and the divergence theorem, we obtain

$$\nu \int_{\mathcal{V}} |\delta \mathbf{G}|^2 dV = - \int_{\mathcal{V}} \delta \dot{\mathbf{x}}' \cdot \delta \dot{\mathbf{x}} dV + \int_{\mathcal{V}} [(\delta \mathbf{G})\delta \dot{\mathbf{x}}] \cdot \dot{\mathbf{x}} dV. \tag{8.13-28}$$

Using the arithmetic-geometric inequality, the Schwarz inequality, and the bound $(17)_1$, we arrive at the statement

$$\int_{\mathcal{V}} |\delta \mathbf{G}|^2 dV \leq -\frac{2}{\nu}\dot{E} + \frac{2N^2}{\nu^2}E. \tag{8.13-29}$$

Substituting (28) into (27) and using (17) yields

$$E\ddot{E} - \dot{E}^2 \geq \frac{2}{\nu}(M^2+1)E\dot{E} - N^2[2(M^2+1)\frac{1}{\nu^2} + 1]E^2. \tag{8.13-30}$$

If

$$K_1 := \frac{2}{\nu}(M^2+1),\ K_2 := N^2 \left[\frac{2}{\nu^2}(M^2+1) \right], \tag{8.13-31}$$

it follows that

$$E\ddot{E} - \dot{E}^2 \geq K_1 E\dot{E} - K_2 E^2. \tag{8.13-32}$$

A theorem of uniqueness established by Serrin[18] states that if E vanishes at any time, it has to vanish for all times. If E vanishes, there is nothing to prove. If E never vanishes, we may divide (31) by E, obtaining (23).

To complete the proof, we observe that (25) can be integrated twice to yield a bound for E.

EXERCISE 8.13.3 (Knops and Payne)

$$E \leq e^{(K_2/K_1)^2}[t + T(1-\lambda)][E(-T)]^{1-\lambda}[E(0)]^{\lambda},$$
$$\lambda := \frac{e^{K_1 t} - e^{-K_1 T}}{1 - e^{-K_1 T}}. \tag{8.13-33}$$

[18] J. Serrin, "The initial value problem for the Navier-Stokes equations," in *Proceedings of the Symposium on Non-Linear Problems*, (University of Wisconsin, 1963), 69–98

Observing that the term within braces on the right-hand side of (26) is nonnegative, applying the Schwarz inequality and the arithmetic-geometric mean inequality, and using the bounds (17), we show that

$$E\ddot{E} - \dot{E}^2 \geq -\tfrac{1}{8}\Bigg\{4M^2 \int_{\mathcal{V}} |\delta\dot{\mathbf{x}}|^2 dV \int_{\mathcal{V}} |\delta\mathbf{G}|^2 dV + N^2\Big(\int_{\mathcal{V}} |\delta\dot{\mathbf{x}}|^2 dV\Big)^2 - \tfrac{1}{4}\int_{\mathcal{V}} |\delta\dot{\mathbf{x}}|^2 dV \Bigg\{\tfrac{1}{2}N^2 \int_{\mathcal{V}} |\delta\dot{\mathbf{x}}|^2 dV + 2\int_{\mathcal{V}} |\delta\mathbf{G}|^2 dV\Bigg\} \qquad (8.13\text{-}34)$$

Finally, we observe that by virtue of the bounds (17)

$$E(-T) \leq 2(M^2 + N^2)\,\mathrm{Vol}(\mathcal{V}), \qquad (8.13\text{-}35)$$

in which $\mathrm{Vol}(\mathcal{V})$ is an upper bound for the volume in $[-T, 0]$ and

$$2E(0) = \int_{\mathcal{V}} |\delta\dot{x}_0|^2 dV. \qquad (8.13\text{-}36)$$

The theorem follows from (33), (34), and (35).

The four preceding theorems rest upon the properties of E as defined by (7). We now exemplify a more general definition of E, namely,

$$E := \int_{\mathcal{V}} g u^2 dV, \qquad (8.13\text{-}37)$$

in which g is positive *weight function* and $\mathbf{u}$ is some kind of perturbance that need not be $\delta\dot{\mathbf{x}}$ and u^2 denotes $|\mathbf{u}|^2$. The method, or the class of methods, is called the *weighted energy method*,[19] rather confusing because, while energy has a definite meaning in mechanics, here "energy" is a mere word, not even defined. Such techniques are often applied to unbounded domains, especially in regard to flows past obstacles. These offer great mathematical difficulty.

To conclude this section, we state a theorem of uniqueness for such flows and outline its proof by the method of weighed energy. In contrast with many earlier theorems, it does not presume much regularity at ∞. In particular, $\mathbf{G}$ need not be bounded, and φ may approach its limit at ∞ rather slowly. We shall use r to denote the distance of $\mathbf{x}$ from some fixed point. A suitable choice of g and u^2, combined with development of the attributes of the particular problem considered, may lead to a bound for $\dot{E}$:

$$\dot{E} \leq kE + F, \qquad (8.13\text{-}38)$$

where k is a positive constant and F a function of t that is in some sense small. We then integrate (38) to yield a bound for E in terms of $E(0)$.

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[19]Expositions of the method are found in the notes of G.P. Galdi and S. Rionero, *Weighted Energy Methods in Fluid Dynamics and Elasticity*, Lecture Notes in Mathematics, (Berlin: Springer-Verlag, 1985).

THEOREM 8.11

Forward uniqueness, initial-value problem in unbounded domains (Rionero and Galdi).[20] *In the time interval [0,T], let* $\dot{\mathbf{x}}$ *and* $\dot{\varphi}$ *satisfy Navier's equation in a domain* $\mathcal{V}$ *exterior to a fixed region* $\mathcal{V}_0$ *containing a unit ball. Let the following regularities be assumed in* $\mathcal{V} \times [0, T]$:

(i) $\dot{\mathbf{x}}$ *is bounded uniformly in t; it and its first derivatives are continuous, it has continuous spatial second derivatives,*

(ii) φ *and* grad φ *are continuous.*

(iii) There are positive constants k, M, *and* r_0 *such that*

$$|\mathbf{G}| \le Mr^k \text{ wherever } r > r_0. \qquad (8.13\text{-}39)$$

(iv) There are constant φ_o *and positive constants* C *and* m, $0 \le m < 1/2$, *such that*

$$|\varphi - \varphi_0| \le \frac{C}{r^{1/2}}(\log r)^{\cdot} \text{ wherever } r > r_0. \qquad (8.13\text{-}40)$$

Then $\dot{\mathbf{x}}$ *is unique.*

Note that (iii) allows **G** *to be unbounded.*

PROOF Considering as before a velocity field $\dot{\mathbf{x}}^*$ and a corresponding pressure field φ^* that satisfy the Navier-Stokes equation in $\mathcal{V} \times [0, T]$, we continue to employ (1) and (2) and the notations defined just after them, except that for conciseness we write $\mathbf{u}$ for the perturbance $\delta\dot{\mathbf{x}}$ and u for $|\mathbf{u}|$. Then selecting some twice differentiable function g and $\mathbf{x}$ alone, we take the scalar product of $g\mathbf{u}$ and (3), obtaining

$$\begin{aligned} \frac{1}{2}(gu^2)' = & -\frac{1}{2}\dot{\mathbf{x}} \cdot \text{grad}(gu^2) + \frac{1}{2}u^2(\dot{\mathbf{x}} \cdot \text{grad } g) \\ & -\frac{1}{2}g\mathbf{u} \cdot \text{grad } u^2 - g\mathbf{u} \cdot [(\text{grad } \dot{\mathbf{x}})\mathbf{u}] \\ & - g[\text{grad}(\delta\varphi)] \cdot \mathbf{u} + \nu g\mathbf{u} \cdot \text{div}\,\delta\mathbf{w}. \end{aligned} \qquad (8.13\text{-}41)$$

Further steps in the proof employ the following transformations and inequalities for solenoidal fields $\mathbf{u}$ and $\mathbf{v}$ and scalars g and f assumed smooth enough for the operations indicated in writing them to be valid:

$$\tfrac{1}{2}(\mathbf{u} \cdot \text{grad } g)u^2 - \tfrac{1}{2}\,\text{div}(gu^2\mathbf{u}) + \tfrac{1}{2}g\mathbf{u} \cdot \text{grad } u^2 = 0, \qquad (8.13\text{-}42)$$

$$g[(\text{grad}\,\mathbf{u})\mathbf{v}] \cdot \mathbf{u} + (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \cdot \text{grad } g + g[(\text{grad } \mathbf{v})\mathbf{u}] \cdot \mathbf{u} - \text{div}[g(\mathbf{u} \cdot \mathbf{v})\mathbf{u}] = 0, \qquad (8.13\text{-}43)$$

[20]For greater detail, see the paper by S. Rionero and G.P. Galdi, "On the uniqueness of viscous fluid motions," *Archive for Rational Mechanics and Analysis* 62 (1976): 295–301.

$$g\mathbf{u} \cdot [(\text{grad } \mathbf{v})\mathbf{u}] \le g\left(\tfrac{u^2 v^2}{2\xi} + \tfrac{\xi g}{2}\right) |\text{grad } \mathbf{u}|^2 \text{ if } \xi > 0, \tag{8.13-44}$$

$$g\Delta\mathbf{u} \cdot \mathbf{u} + g|\text{grad } \mathbf{u}|^2 - \tfrac{1}{2}u^2 \Delta g - \tfrac{1}{2}\,\text{div}(g \text{ grad } u^2 - u^2 \text{ grad } g) = 0, \tag{8.13-45}$$

$$(\delta f)\,\text{grad } g \cdot \mathbf{u} - \text{div}[(\delta f)\mathbf{u}] + g[\text{grad}\,(\delta f)] \cdot \mathbf{u} = 0. \tag{8.13-46}$$

In the last statement, f is a sufficiently smooth scalar field.

We now denote by S_R a ball of radius R containing $\mathcal{V}_0$, and $\mathcal{V}_R := \mathcal{V} \cap S_R$. Integrating (40) over the domain $\mathcal{V}_R$ and the relations (41)–(45), we obtain for all positive ξ

$$\begin{aligned}
&\left(\frac{1}{2}\int_{\mathcal{V}_R} g u^2 dV\right)' \\
&\quad \le \int_{\mathcal{V}_R} \left\{ \frac{1}{2}u^2(\mathbf{u} \cdot \text{grad } g) + (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \cdot \text{grad } g \right. \\
&\quad + \frac{1}{2}u^2(\mathbf{v} \cdot \text{grad } g) + g\frac{u^2\mathbf{v}^2}{2\xi} + (\delta\varphi)\mathbf{u} \cdot \text{grad } g + \frac{1}{2}\nu u^2 \Delta g \\
&\quad \left. + \left(\frac{1}{2}\xi - \nu\right) g|\text{grad } u^2| \right\} dV + \int_{\partial\mathcal{V}_R} \left\{ \frac{1}{2}\nu \text{ grad}\,(g u^2) - \nu u^2 \text{ grad } g \right. \\
&\quad \left. - \frac{1}{2}g u^2 \mathbf{u} - g(\mathbf{u} - \mathbf{v})\mathbf{u} - g(\delta\varphi)\mathbf{u} + \frac{1}{2}g u^2 \mathbf{v} \right\} \cdot \mathbf{n}\, dA.
\end{aligned} \tag{8.13-47}$$

Choosing

$$g = e^{-mr^\alpha}, \xi = 2\nu, m \ge 4, \alpha \epsilon (0, 1), \tag{8.13-48}$$

Rionero and Galdi show that as $R \to \infty$, the inequality (46) reduces to

$$\dot{E} \le kE + m\alpha \int_{\mathcal{V}} r^{\alpha-1} g|\delta\varphi||\delta\dot{\mathbf{x}}|dV, \tag{8.13-49}$$

$$k := \sup\left\{ m|\mathbf{u}|, 3m|\mathbf{v}|, \frac{V^2}{2v}, \frac{v}{2}m(m+2) \right\}. \tag{8.13-50}$$

It follows from the Cauchy inequality that

$$\alpha r^{\alpha-1}|\delta\varphi|u \le \frac{1}{2}\alpha^2 r^{2(\alpha-1)}|\delta\varphi|^2 + \frac{1}{2}u^2, \tag{8.13-51}$$

and thus

$$\dot{E} \le k_1 E + \frac{m\alpha^2}{\rho}\int_{\mathcal{V}} r^{2(\alpha-1)} g|\delta\varphi|^2 dV, k_1 := k + \frac{m}{\rho}. \tag{8.13-52}$$

By (50),

$$\dot{E} \le k_1 E + \frac{m\alpha^2}{\rho} \int\limits_{\mathcal{V}_R} |\delta p|^2 dV + \frac{m\alpha^2}{2\rho} \int\limits_{\mathcal{V}-\mathcal{V}_R} r^{2(\alpha-1)} g |\delta p|^2 dV$$

$$\le k_1 E + k_2 \alpha^2 + \frac{\lambda^2 m \alpha^2 2\pi}{\rho} \int\limits_1^\infty r^{2\alpha-1} e^{-mr^\alpha} (\log r)^{2\sigma} dr, \qquad (8.13\text{-}53)$$

in which

$$k_2 := \frac{m}{2\rho} \sup \int\limits_{\mathcal{V}_R} |\delta p|^2 dV. \qquad (8.13\text{-}54)$$

Since $m \ge 4$, we find that

$$\dot{E} \le k_1 E + F(\alpha), F(\alpha) := k_2 \alpha^2 + \alpha^{1-2\sigma} \left(\frac{4\pi^2 \lambda^2 m}{2\rho} \right) (2\sigma)^{2\sigma} e^{-2\sigma}. \qquad (8.13\text{-}55)$$

Integrating (53) from 0 to τ, and observing that $E(0)$=0, we obtain

$$E(\tau) \le F_1(\alpha) \tau e^{k_1 T} \le F_1(\alpha) T e^{k_1 T}, (T \ge \tau). \qquad (8.13\text{-}56)$$

It follows from (36) that

$$E(\tau) \ge \int\limits_{\mathcal{V}_\delta} u^2 e^{-mr^\alpha} dV \ge e^{-m\delta^\alpha} \int\limits_{\mathcal{V}_\delta} u^2 dV, \qquad (8.13\text{-}57)$$

where

$$\mathcal{V}_\delta := \mathcal{S}_\delta \cap \mathcal{V}, \qquad (8.13\text{-}58)$$

and hence, since $\alpha \in (0,1)$,

$$\int\limits_{\mathcal{V}_\delta} u^2 dV \ge e^{m\delta} F_1(\alpha) T e^{kT_1}. \qquad (8.13\text{-}59)$$

Since $\lim_{a \to 0} e^{m\delta} F_1(\alpha) T e^{k_1 T} = 0$, it immediately follows that the solution is unique, for if it were otherwise, continuity of $\mathbf{u}$ would imply the existence of a positive number E such that

$$\int\limits_{\mathcal{V}_\delta} u^2 dV \le E, \qquad (8.13\text{-}60)$$

which is a contradiction.

Galdi and Padula[21] have recently constructed a broad extension of Serrin's analysis of stability. Not only do they obtain by an energy method weaker sufficient

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[21] G.P. Galdi and M. Padula, "A new approach to energy theory in the stability of fluid motion," *Archive for Rational Mechanics and Analysis* 110 (1990): 187–286.

conditions for stability, but they carry through their analysis within a frame more general than that of the Navier-Stokes theory.

8.14 Instability

For a long time that theory was thought able to deliver only sufficient conditions for stability, but, as students who persevere to the end of this section will learn, a suitable approach within the rigorous theory also yields conditions sufficient for instability.

Galdi and Padula set up equations of the form

$$\mathbf{B}\mathbf{u}_t = \mathbf{L}\mathbf{u} + \mathbf{N}\mathbf{u}, \;\; \mathbf{u}(0) = \mathbf{u}_0, \tag{8.14-1}$$

in which $\mathbf{u}$ belongs to an appropriate Hilbert space with some specificable structure of regularity. $\mathbf{B}$ is a positive diagonal, linear transformation; $\mathbf{L}$ is a linear operator that may be unbounded; and $\mathbf{N}$ is an operator, perhaps nonlinear, that satisfies the condition

$$\mathbf{N}(0) = 0. \tag{8.14-2}$$

We observe that if

$$\begin{aligned} \mathbf{u} &:= \delta\dot{\mathbf{x}}, \quad \mathbf{B} := 1, \\ \mathbf{L}\delta\dot{\mathbf{x}} &:= 2\nu \;\operatorname{div}\delta\mathbf{w} - \operatorname{grad}\;\delta\varphi, \\ \mathbf{N}\delta\dot{\mathbf{x}} &:= \mathbf{G}\delta\dot{\mathbf{x}} + (\delta\mathbf{G})\dot{\mathbf{x}}^*, \end{aligned} \tag{8.14-3}$$

then (1) reduces to the equation that governs the perturbance of a basic solution of Navier'S dynamical equation. The null solution $\mathbf{u} = \mathbf{0}$ satisfies (1) if $\mathbf{u}_0 = \mathbf{0}$.

The stability or instability of the flow depends upon the properties of $\mathbf{L}$ and $\mathbf{N}$. We call $\mathbf{L}$ *essentially dissipative* if

a. $(\mathbf{L}\mathbf{w}, w) \leq 0$ for all $\mathbf{w}$ in the specified Hilbert space,

b. $(\mathbf{L}\mathbf{w}, w) = 0 \Rightarrow \mathbf{w} = 0$.

Here (.,.) is the inner product in the Hilbert space selected. Galdi and Padula prove that if $\mathbf{L}$ is symmetric and essentially dissipative, and if $\mathbf{N}$ is suitable in the sense that $\|\mathbf{N}\mathbf{u}\|$ is bounded in an appropriate manner, where $\|\cdot\|$ is the norm induced by the inner product, then the null solution is monotonically stable in the energy $(\mathbf{B}\mathbf{u}, u)$. They show also that if there is a perturbance $\mathbf{w}$ that belongs to the Hilbert space and is such that $(\mathbf{L}\mathbf{w}, w,) > 0$, and if $\|\mathbf{N}\mathbf{u}\|$ is bounded appropriately, then the null solution is unstable.

The stability theorem extends Serrin's method within the framework of Sobolev spaces. While intricate inequalities are used, the method is much like Serrin's.

On the other hand, the theorem of instability is altogether new, and we shall sketch a proof of it, making a special choice of **N**.

Replacing **u** by $\delta\dot{\mathbf{x}}$, from (1) we see that

$$\delta\dot{\mathbf{x}}' = \mathbf{B}^{-1}\mathbf{L}\delta\dot{\mathbf{x}} + \mathbf{B}^{-1}\mathbf{N}\delta\dot{\mathbf{x}}, \tag{8.14-4}$$

and thus

$$(\delta\dot{\mathbf{x}}', \mathbf{L}\delta\dot{\mathbf{x}}) = (\mathbf{B}^{-1}\mathbf{L}\delta\dot{\mathbf{x}}, \mathbf{L}\delta\dot{\mathbf{x}}) + (\mathbf{B}^{-1}\mathbf{N}\delta\dot{\mathbf{x}}, \mathbf{L}\delta\dot{\mathbf{x}}). \tag{8.14-5}$$

In the special case $\mathbf{N} \equiv \mathbf{0}$, which satisfies (2), (4) reduces to

$$(\delta\dot{\mathbf{x}}', \mathbf{L}\delta\dot{\mathbf{x}}) = (\mathbf{B}^{-1}\mathbf{L}\delta\dot{\mathbf{x}}, \mathbf{L}\delta\dot{\mathbf{x}}), \tag{8.14-6}$$

and thus

$$\frac{1}{2}(\delta\dot{\mathbf{x}}', \mathbf{L}\delta\dot{\mathbf{x}})' = (\mathbf{B}^{-1}\mathbf{L}\delta\dot{\mathbf{x}}, \mathbf{L}\delta\dot{\mathbf{x}}) > 0, \tag{8.14-7}$$

in which the last inequality is true because **B** is positive. Next, forming the scalar product of (1) and $\delta\dot{\mathbf{x}}$, we obtain

$$\frac{1}{2}(\mathbf{B}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}})' = (\mathbf{L}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}}) + (\mathbf{N}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}}), \tag{8.14-8}$$

which if $\mathbf{N} \equiv \mathbf{0}$ reduces to

$$\frac{1}{2}(\mathbf{B}\delta\mathbf{x}, \delta\dot{\mathbf{x}})' = (\mathbf{L}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}}). \tag{8.14-9}$$

Suppose there is a perturbance such that $(\mathbf{Lw}, \mathbf{w}) > 0$, and we make the choice $\delta\dot{\mathbf{x}}(0) = \mathbf{w}$. Then (6) implies that

$$(\mathbf{L}\delta\mathbf{x}, \delta\dot{\mathbf{x}}) > (\mathbf{L}\delta\dot{\mathbf{x}}(\mathbf{0}), \delta\dot{\mathbf{x}}(0)) > 0, \tag{8.14-10}$$

which, when used in conjunction with (10), yields

$$(\mathbf{B}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}}) \geq (\mathbf{B}\delta\dot{\mathbf{x}}(0), \delta\dot{\mathbf{x}}(0)) + 2(\mathbf{L}\delta\dot{\mathbf{x}}(0), \delta\dot{\mathbf{x}}(0))t. \tag{8.14-11}$$

It follows immediately that

$$(\mathbf{B}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}}) \to \infty \text{ as } t > 0. \tag{8.14-12}$$

Thus, finally we have shown that if there is a **w** such that $(\mathbf{Lw}, \mathbf{w}) > 0$, then the flow is unstable.

If $\mathbf{N} \equiv \mathbf{0}$, the proof of instability is very involved.

More definite conclusions regarding stability can be obtained if we have more information about the structure of **L** and **N**. In many applications,

$$\begin{gathered} \mathbf{L} = \mathbf{L}_s + \boldsymbol{\lambda}_1\mathbf{M} + \boldsymbol{\lambda}_2\mathbf{P},\ \boldsymbol{\lambda}_2 > 0, \\ (\mathbf{L}_s\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{L}_s\mathbf{v}), \\ (\mathbf{Mu}, \mathbf{v}) = -(\mathbf{u}, \mathbf{Mv}), \end{gathered} \tag{8.14-13}$$

and $\mathbf{P}$ is a bounded operator that reduces to the null operator if the basic flow is the null state. Because $\mathbf{M}$ is skew, it follows from (7) that

$$(\mathbf{B}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}})' = (\mathbf{L}_s\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}}) + (\mathbf{N}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}}). \tag{8.14-14}$$

If $\mathbf{N} \equiv \mathbf{0}$, the evolution of $(\mathbf{B}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}})$ is completely determined by the symmetric, linear operator $\mathbf{L}_s$. Galdi and Padula present examples that show that here lies the root of the extremely low estimates of stability limits delivered by the classical analysis of Serrin and others who used $(\mathbf{B}\delta\dot{\mathbf{x}}, \delta\dot{\mathbf{x}})$ as their measure of energy. To get better estimates for the onset of instability requires a more general functional that incorporates information about the skew operator $\mathbf{M}$. Much of the paper of Galdi and Padula is devoted to showing how such functionals can be constructed.

8.15 Some Unsteady Flows

The simplest unsteady flows of Navier-Stokes fluids were obtained for instances in which the vorticity equation (8.2-2) reduces to

$$\mathbf{W}' = 2\nu \operatorname{skw} \operatorname{grad} \operatorname{div} \mathbf{W}, \tag{8.15-1}$$

each Cartesian component of which has the form of the equation that governs diffusion of heat in a rigid, isotropic conductor. An example is provided by plane flows such that $\mathbf{W}$ is constant on each streamline at every time; for such a flow we may put $\alpha_1 = 0$ in (7.3-3) and so obtain the scalar equation

$$w' = \nu\Delta w + \dot{\mathbf{x}} \cdot \operatorname{grad} \dot{\mathbf{x}} = 0. \tag{8.15-2}$$

Also in flows such that $\dot{\mathbf{x}} \cdot \operatorname{grad} \dot{\mathbf{x}} = \mathbf{0}$ and $\mathbf{b} - \nu \operatorname{grad} p$ is perpendicular to the plane $x = 0$, (7.2-1) reduces essentially to

$$\dot{x}' = \nu\Delta\dot{x}, \tag{8.15-3}$$

again the equation governing flow of heat in one dimension. Equations of this kind are the simplest of linear parabolic partial differential equations, and a great corpus of mathematical theory regarding them is available. We shall discuss some instances that offer interest in the context of Navier-Stokes flows. Student's must take care not to presume that the properties of any linear partial differential equation can reveal much about the Navier-Stokes theory.

8.16 Some Oscillatory Flows

In Section 7.2, we studied several flows induced in bodies of fluid of grade 2 by the oscillation of a plate or by the relative motion of two parallel plates. As the special instances in which $\alpha_1 = \alpha_2 = 0$, we may obtain solutions for the Navier-Stokes theory, and some of these we have already remarked and discussed.

We now derive a solution due to Stokes that delivers azimuthal oscillations in a body of Navier-Stokes fluid surrounding an infinitely long cylindrical rod of radius r_0 that is compelled to oscillate at the assigned frequency α. To that end we introduce the following contravariant components of velocity in cylindrical polar coordinates:

$$\dot{r} = 0, \quad \dot{q} = \omega(r, t), \quad \dot{z} = 0, \tag{8.16-1}$$

and we assume that $\mathbf{b} - \text{grad } p/\rho$ lies in the plane $\theta = $ constant at each point.

Putting (1) into (8.15-2) gives us

$$\partial_t \omega = \nu \left(\partial_{rr}\omega + \frac{1}{r}\partial_r \omega - \frac{\omega}{r^2} \right). \tag{8.16-2}$$

We seek a solution such that

$$\begin{gathered} \omega(r_0, t) = V \cos(\alpha t),\ V = \text{const.}, \\ \omega(r, t) \to 0 \text{ as } r \to \infty \text{ when } t = \text{const.} \end{gathered} \tag{8.16-3}$$

For a separable solution

$$\omega(r, t) = W(r)T(t), \tag{8.16-4}$$

it is necessary that

$$\begin{gathered} T' - \lambda T = 0, \lambda = \text{const.}, \\ W'' + \frac{1}{r}W' - \left(\frac{1}{r^2} + \frac{\lambda}{\nu} \right) W = 0. \end{gathered} \tag{8.16-5}$$

In terms of the new variables defined by

$$s \equiv \left(\frac{\lambda}{\nu} \right)^{1/2}, Y(s) \equiv W(r), \tag{8.16-6}$$

$(5)_2$ takes the form

$$s^2 Y'' + Y' - (1 + s^2)Y = 0. \tag{8.16-7}$$

It follows from (3)–(7) that

$$w(r, t) = Re \left\{ \frac{K_1 \left[\left(\frac{i\alpha}{\nu} \right)^{1/2} r \right]}{K_1 \left[\left(\frac{i\alpha}{\nu} \right)^{1/2} r_0 \right]} V e^{i\alpha t} \right\}, \tag{8.16-8}$$

in which K_1 denotes a Bessel function in standard notation and Re means "real part of."

EXERCISE 8.16.1
Show that the only solution of (2) that satisfies (3) and (4) is given by (8). As (8.6-2) is linear, solutions may be superposed. An example follows.

EXERCISE 8.16.2 (Casarella and Laura)
Assuming that $\mathbf{b} - \text{grad}\, p/\rho$ is radial, consider the velocity field

$$\dot{r} = 0, \dot{\theta} = w(r, t), \dot{z} = u(r, t) \tag{8.16-9}$$

with the boundary conditions (3) and

$$u(r_0, t) = U\cos(\beta t), u(r, t) \to 0 \quad \text{as} \quad r \to \infty \quad \text{when } t = \text{const.} \tag{8.16-10}$$

Show that (8) holds and also

$$u = Re\left\{\frac{K_0\left[\left(\frac{i\beta}{\nu}\right)^{1/2} r\right]}{K_0\left[\left(\frac{i\beta}{\nu}\right)^{1/2} r_0\right]} U e^{i\beta t}\right\}, \tag{8.16-11}$$

where K_0 is a Bessel function in standard notation.

8.17 Flow Due to a Plane Boundary Moved Suddenly from Rest

Consider a plane at rest, above which rests a semi-infinite region of Navier-Stokes fluid, made to move suddenly with speed U in its own plane. We shall assume that the velocity field has the following form in a rectangular Cartesian coordinate system

$$\dot{x}_1 = u(x_2, t), \dot{x}_2 = 0, \dot{x}_3 = 0. \tag{8.17-1}$$

Substituting (1) into (8.1-1) and writing x for x_2, we obtain

$$u' = \nu \frac{\partial^2 u}{\partial x^2}. \tag{8.17-2}$$

In deriving (2) we have assumed that the pressure and the body force field are independent of x_1. The appropriate initial and boundary conditions that govern the problem are

$$u(x, 0) = 0, \quad \forall x > 0, \tag{8.17-3}$$

$$u(0, t) = U, \quad \forall t > 0. \tag{8.17-4}$$

An analogous problem governs the conduction of heat and has been studied in detail. The solution is given by

$$u(x, t) = U \left\{ 1 - erf \left[\frac{x}{2(\nu t)^{1/2}} \right] \right\}, \tag{8.17-5}$$

where

$$erf\, \alpha := \frac{2}{\sqrt{\pi}} \int_0^2 e^{-\alpha^2} d\alpha. \tag{8.17-6}$$

EXERCISE 8.17.1

Verity that the statement (5) is the solution of (2) subject to (3) and (4).

Rayleigh recognized that the foregoing solution provides a very simple and elegant analogy by means of which we can understand the boundary-layer development due to the steady flow past an infinite flate plate. If one assumes that the disturbance due to the presence of the plate diffuses at the rate given by the unsteady problem while simultaneously moving downstream with the velocity U, then at a distance d from the leading edge of the plate, the boundary-layer thickness, by analogy, will be $\sqrt{\nu d/U}$.

Detailed solutions for the equation (2) for a variety of initial and boundary conditions can be found in the article by Berker cited at the end of the chapter. The following exercises provide a few important examples.

EXERCISE 8.17.2

Determine the general solution of (2) subject to

$$u(x)_2, 0) = f(x_2)\, \forall \quad 0 < x_2 < \infty, \; u(0, t) = 0 \quad \forall\, 0 < t < \infty \tag{8.17-7}$$

where $f(x_2)$ is an assignable function.

EXERCISE 8.17.3

Determine the general solution to (15) subject to

$$u(x_2, 0) = 0, \; \forall\, 0 < x_2 < \infty, \; u(0, t) = g(t), \quad \forall\, 0 < t < \infty \tag{8.17-8}$$

where $g(t)$ is an assignable function.

8.18 Flow Due to the Sudden Application of a Pressure Gradient

We shall now consider the flow of a Navier-Stokes fluid within a pipe of circular cross-section by suddenly applying a constant pressure gradient along the axis of the pipe. We assume that the contravariant components of the velocity field in a cylindrical coordinate system are given by

$$\dot{r} = 0, \dot{\theta} = 0 \quad \text{and } \dot{z} = u(r, t). \tag{8.18-1}$$

Substituting (2) into (8.1-1) yields

$$u' = \frac{C}{\rho} + \nu \left(u_{rr} + \frac{1}{r} u_r \right), \tag{8.18-2}$$

in which the constant C is the value of the pressure gradient that is maintained in the axial direction. The appropriate boundary condition is

$$u(r_0, t) = 0, \ 0 \leq t < \infty, \tag{8.18-3}$$

where r_0 is the radius of the pipe. The initial condition is

$$u(r, t) = 0, \ 0 \leq r < r_0. \tag{8.18-4}$$

Use of a new variable $v(r, t)$

$$v = \frac{c}{4\mu}(r_0^2 - r^2) - u \tag{8.18-5}$$

gives (2) the form

$$v' = \nu \left(\partial_{rr} v + \frac{1}{r} \partial_r v \right), \tag{8.18-6}$$

with

$$v(r_0, t) = 0, \ 0 \leq t < \infty, \tag{8.18-7}$$

$$v(r, 0) = \frac{C}{4\mu}(r_0^2 - r^2), \ 0 \leq r \leq r_0. \tag{8.18-8}$$

EXERCISE 8.18.1

Use separation of variables to show that the solution of (6) subject to (7) and (8) is given by

$$v(r, t) = \frac{C}{\mu} \sum_{n=1}^{\infty} C_n J_0 \left(\lambda_n \frac{r}{r_0} \right) \exp \left(- \left(\frac{\lambda_n}{r_0} \right)^2 \nu t \right), \tag{8.18-9}$$

J_0 being a Bessel function the positive zeroes of which are $\lambda_n, n = 1, 2, 3, \dots$, and

$$C_n = \frac{8r_0^2}{\lambda_n^3 J_1(\lambda_n)}, n = 1, 2, 3, \dots , \tag{8.18-10}$$

where J_1 is a Bessel function.

It follows from (5) and (9) that the speed u is given by

$$u(r, t) = \frac{c(r_0^2 - r^2)}{4\mu} - \frac{2cr_0^2}{\mu} \sum_{n=1}^{\infty} \frac{J_0\left(\lambda_n \frac{r}{r_0}\right)}{\lambda_n^3 J_1(\lambda_n)} \exp\left[-\left(\frac{\lambda_n}{r_0}\right)^2 \nu t\right]. \tag{8.18-11}$$

A detailed discussion of how the velocity profile is affected by the material parameters can be found in the article by Berker.[22] Initially, the retarding effect of the wall is felt in a narrow region adjacent to the wall; the effect spreads further inward with time and after a critical time t_{cr} the effect of the wall is felt everywhere and the velocity approaches the steady value.

8.19 Unsteady Flow Impinging on a Flat Plate

In Section 6, we studied the steady flow of a Navier-Stokes fluid that impinges on an infinite flat plate. Guided by the work there, we shall now study an unsteady contribution to (8.6-2), using as a starting point for semiinverse consideration for the class of isochoric velocity fields

$$\begin{aligned} \dot{x}_1 &= x_1 f'(x_2) + e^{i\omega t} g(x_2), \\ \dot{x}_2 &= \; - f(x_2), \dot{x}_3 = 0, \end{aligned} \tag{8.19-1}$$

under the assumption that p is steady. The velocity field (1) could, for instance, describe the situation of the fluid impinging on a flat surface that is stretching unsteadily. The case $\dot{x}_3 = 0$ corresponds to the problem wherein the flat boundary is stretched steadily and has been studied in detail by Danberg and Fansler.

[22] R. Berker, "Intégration des équations du mouvement dún fluide visqueux incompressible," in *Handbuch der Physik*, 8_2, edited by S. Flügge and C. Truesdell (Berlin, Göttingen, and Heidelberg: Springer-Verlag, 1963).

EXERCISE 8.19.1
Show that the differential equations of motion deliver the same conditions on f as in Section 6 and on g simply

$$\nu g'' + fg' - f'g + i\omega g = 0. \tag{8.19-2}$$

If the infinite plate is impervious and stretches periodically with frequency ϖ, then (2) can be appropriately nondimensionalized so that $g(0) = 1$, and for the flow to approach its irrotational counterpart at ∞, it is necessary that

$$g' \to 0 \text{ as } x_2 \to \infty. \tag{8.19-3}$$

We note that the conditions applied to g are linear in g. Thus, if f has been determined as a solution of the problem discussed in Section 6; we can then determine g with relative ease. For the case $\omega = 0$, Danberg and Fansler determine an explicitly exact solution for g by using the method of variation of parameters.

For the flow field (1), the full Navier-Stokes equations and the boundary-layer approximations yield the same equations. The velocity field (1) allows us to consider flows that at $x_2 = \infty$ move unsteadily along the x_1 direction. Such flows have relevance to important practical problems in aerodynamics and have been studied numerically in great detail by Rott and Glaubert.

8.20 Linearly Polarized Waves

We next consider the possibility of propagation of linearly polarized waves in a Navier-Stokes fluid. Let us consider the velocity field

$$\dot{x}_1 = 0, \quad \dot{x}_2 = 0, \quad \dot{x}_3 = w(x_1, x_2, t). \tag{8.20-1}$$

Clearly, this above velocity field is isochoric.

EXERCISE 8.20.1 (Boulanger, Hayes, and Rajagopal[23])
Verify that

$$\begin{aligned} w(x_1, x_2, t) = {} & A\cos\{Kmx_1\cos\psi - Kx_2\sin\psi - \Omega t\sin 2\psi\} \\ & \exp\{-Kmx_1\sin\psi - Kx_2\cos\psi - \Omega t 2\psi\}, \\ \psi = {} & \text{constant}, \end{aligned} \tag{8.20-2}$$

[23] P. Boulanger, M. Hayes, and K.R. Rajagopal, "Some unsteady exact solutions in the Navier-Stokes and second-grade fluid theories," *Stability and Applied Analysis of Continua* 1 (1991): 185–204.

satisfy the Navier-Stokes equation, where A, Ω, ψ, and m are arbitrary constants, $m \neq 1$, and

$$K^2 := \frac{\rho\Omega}{\mu(m^2-1)}. \tag{8.20-3}$$

The flow field (1) represents an unhomogeneous finite amplitude wave linearly polarized along the x_3 axis. The phase is propagated unchanged along the direction $m\cos\varphi\mathbf{i} - \sin\varphi\mathbf{j}$, its velocity $\mathbf{v}_p$ being given by

$$\mathbf{v}_p = \frac{\Omega\sin 2\psi m\cos\psi\mathbf{i} - \sin\psi\mathbf{j}}{K(m^2\cos^2\psi - \sin^2\psi)}. \tag{8.20-4}$$

The amplitude is propagated unchanged along $-m\cos\varphi\mathbf{i} - \sin\varphi\mathbf{j}$ with velocity $\mathbf{v}_a$ given by

$$\mathbf{v}_a = \frac{\Omega\cos 2\psi(m\sin\psi\mathbf{i} - \cos\psi\mathbf{j})}{K(m^2\sin^2\psi - \cos^2\psi)}. \tag{8.20-5}$$

The angle θ between the planes of equal phase and the planes of equal amplitude is given by

$$\tan\theta = \frac{2m}{(m^2-1)\sin\psi}. \tag{8.20-6}$$

The case $\psi = 0$ is of special interest, for we have a nonpropagating solution, *i.e.*,

$$\dot{x}_1 = 0, \quad \dot{x}_2 = 0, \quad \dot{x}_3 = A\cos kmx_1\exp\{-(kx_2 + \Omega t)\}, \tag{8.20-7}$$

where

$$k = \frac{\rho\Omega}{\mu(m^2-1)}. \tag{8.20-8}$$

The angle between the planes of equal phase and the planes of equal magnitude in this case is $\frac{1}{2}\pi$.

Similarly, when $\psi = \frac{1}{2}\pi$ we have another elegant nonpropagating solution

$$\dot{x}_1 = 0, \dot{x}_2 = 0, \dot{x}_3 = A\cos k\,x_2\,\exp\{-kmx_1 + \Omega t\}. \tag{8.20-9}$$

The possibility of circular polarized waves and other unsteady flows of the form (1) in both the Navier-Stokes fluid and the fluids of grade 2 are discussed at length by Boulanger, Hayes, and Rajagopal.

8.21 Further Unsteady Plane Flows

We shall now study a class of unsteady flows that includes as special cases flows that Berker classified as "Jeffrey motions" and "Taylor flows." Consider the isochoric velocity field

$$\dot{x}_2 = x_1 \frac{\partial f}{\partial x_2}, \quad \dot{x}_2 = -f(x_2, t), \quad \dot{x}_3 = 0. \tag{8.21-1}$$

EXERCISE 8.21.1

Show that the differential equations governing the vorticity lead to

$$\mu \frac{\partial^4 f}{\partial x_2^4} - \rho \frac{\partial^3 f}{\partial x_2^2 \partial t} = \rho \frac{\partial}{\partial x_2} \left\{ \left(\frac{\partial f}{\partial x_2} \right)^2 - f \frac{\partial^2 f}{\partial x_2^2} \right\}. \tag{8.21-2}$$

Before discussing any specific initial boundary value problem, we observe that for flows in which the vorticity is constant along the streamlines, the right-hand side of (2) is zero.

We notice that

$$f(x_2, t) = [a(t)]x_2 + b(t), \tag{8.21-3}$$

where $a(t)$ and $b(t)$ are continuously differentiable but otherwise arbitrary functions of time, satisfies (2). The flow corresponding to (3) is irrotational and belongs to the class of flows classified as "Jeffrey motions" by Berker.

EXERCISE 8.21.2

Consider the flow of a Navier-Stokes fluid between two infinite parallel elastic sheets coinciding with the planes $x_2 = \frac{(\pi/2-d)}{k}$ and $x_2 = \frac{-(\pi/2+d)}{k}$, where d and k are nonzero constants. Suppose the stretching motions of the elastic sheet are given by

$$\dot{x}_1 \left(x_1, \frac{\pi - 2d}{2k}, t \right) = kCx_1 \exp\{-\Omega t\},$$
$$\dot{x}_1 \left(x_1, \frac{-(\pi - 2d)}{2k}, t \right) = -kCx_1 \exp\{-\Omega t\}, \tag{8.21-4}$$

where C is a constant. Show that the solution to (2) that meets (4) is

$$f(x_2, t) = -C \exp\{-\Omega t\} \cos(kx_2 + d), \Omega := \frac{\mu k_2}{\rho}. \tag{8.21-5}$$

The flow given by (5) falls under the category that Berker calls "Taylor flows." Several special solutions to (2) that correspond to different physical problems have been discussed by Boulanger, Hayes and Rajagopal.

General References

[1.] Serrin, J. "Mathematical principles of classical fluid mechanics." In *Handbuch der Physik* 8_1. edited by S. Flügge and C. Truesdell. Berlin, Göttingen, and Heidelberg: Springer-Verlag, 1959.

[2.] Berker, R. "Integration des équations du mouvement d'un fluide visqueux incompressible." *Handbuch der Physik* 8_2, edited by S. Flügge and C. Truesdell. Berlin, Göttingen, and Heidelberg: Springer-Verlag, 1963.

9

Incompressible Euler Fluids

9.1 Preliminaries

While students would do well to reread Section 8.1, here for convenience some generalities regarding inviscid fluids are repeated. Many can be obtained formally by simply annulling μ and ν in statements made in Section 8.1–8.4, but because Eulerian hydrodynamics is not only the prototype of continuum mechanics but also its most perfect example, we prefer to repeat its basic qualities here.

An *unconstrained elastic fluid* is defined by the constitutive equation

$$\mathbf{T} = -p(\rho)\mathbf{1}, \tag{4.1-8}$$

in which p is the pressure field. Such a fluid is called *compressible*; in this chapter "compressible fluid" will always refer to an elastic fluid. For an *incompressible elastic fluid* the determinate stress is null, so the constitutive equation is

$$\mathbf{T} = -p\mathbf{1}, \tag{4.1-9}$$

in which p is a scalar field not determined by the motion of the fluid, which must be isochoric: $\operatorname{div} \dot{\mathbf{x}} = 0$. If we substitute either of these two constitutive equations into Cauchy's first law of motion (2.7-5), we obtain *Euler's dynamical equation* for elastic fluids:

$$\rho\ddot{\mathbf{x}} = -\operatorname{grad} p + \rho\mathbf{b}. \tag{9.1-1}$$

Although this equation governs the motion of elastic fluids in general, the student must remember that the symbol p means one thing for a compressible fluid and quite another for an incompressible one. For the former, it is a given function of the density; for the latter, it is a scalar field whose value at a time and place is determined by the history of the motion of the part of the fluid body near that place. Unless the contrary is stated, we continue to suppose the fluid body be homogeneous in that for a compressible fluid the function p in (8) is independent of $\mathbf{X}$, while for an incompressible fluid ρ is an assigned constant, likewise independent of $\mathbf{X}$.

The distinction is important, for in classical hydrodynamics there is an extensive literature on flows of incompressible elastic fluids of nonuniform density. These are often called inhomogeneous flows. In Section 9, we shall consider two particular flows of this kind.

Flows of elastic fluids have an energy integral; here we shall obtain it for flows of compressible fluids as a consequence of a general theorem of hyperelasticity.

Euler noticed that if

$$R := \begin{cases} \int \frac{1}{\rho}\frac{dp}{d\rho}d\rho & \text{for a compressible fluid,} \\ \frac{p}{\rho} & \text{for an incompressible fluid,} \end{cases} \tag{9.1-2}$$

then (1) assumes the form

$$\ddot{\mathbf{x}} = -\operatorname{grad} R + \mathbf{b}, \tag{9.1-3}$$

and he drew a number of simple conclusions from this fact. First, $\mathbf{b} - \ddot{\mathbf{x}}$ is lamellar. Hence *the acceleration field is lamellar if and only if the body-force field is lamellar*. In particular, accelerationless flow is impossible if the body force is not lamellar. *A fortiori, only when subject to a lamellar body force can a fluid remain in equilibrium.*

The boundary condition

$$\mathbf{n} \cdot \dot{\mathbf{x}} = 0, \tag{1.8-5}$$

weaker than the condition of adherence (1.8-7), is appropriate to an Eulerian fluid at a stationary wall. At a free boundary the pressure may be assigned, perhaps as a constant, perhaps as given by a surface tension.[1]

9.2 Compatibility: General Solution of Euler's Dynamical Equation

Henceforth we shall assume that $\mathbf{b}$ is lamellar:

$$\mathbf{b} = -\operatorname{grad} \varpi. \tag{9.2-1}$$

So also, (9.1-3) shows us, is $\ddot{\mathbf{x}}$. Such is the case if and only if the D'Alembert-Euler condition (2.2-25) is satisfied. Works on hydrodynamics usually present that condition in the equivalent form furnished by (2.2-30):

$$\dot{\mathbf{W}} + \mathbf{DW} + \mathbf{WD} = \mathbf{0}. \tag{9.2-2}$$

[1]See section 3.8 of C. Truesdell, *A First Course in Rational Continuum Mechanics*, vol 1. (New York: Academic Press, 1991), for a discussion of this topic.

Equivalently, in a simply connected region there is an acceleration potential $P_{\mathbf{a}}$:

$$\ddot{\mathbf{x}} = -\operatorname{grad} P_{\mathbf{a}}. \tag{9.2-3}$$

Comparison with (1) and (9.1-3) shows that

$$R = P_{\mathbf{a}} - \varpi. \tag{9.2-4}$$

The potential ϖ is usually regarded as an assigned field.

The foregoing statements express *Euler's general solution* of his dynamical equation (9.1-3) when $\mathbf{b}$ is assumed lamellar. It illustrates Leibniz's program: By these few considerations the whole matter is reduced to pure geometry, the one thing to be desired in physics and mechanics." The partial differential equation (2) is both general and purely kinematical. It is a *condition of integrability* for the scalar field $P_{\mathbf{a}}$ in (2.2-29). A velocity field $\dot{\mathbf{x}}$ that satisfies it delivers $\ddot{\mathbf{x}}$ by routine calculation, and from $\ddot{\mathbf{x}}$ the acceleration potential $P_{\mathbf{a}}$ is easy to determine. Then R is given explicitly by (4). The relation (9.1-2) would seem then to deliver the pressure field p. Indeed, if the fluid is homogeneous and incompressible, then (9.1-2) does indeed deliver p.

This ideal program is difficult to carry through in practice, for we generally think of flows as arising from initial conditions and conforming with boundary conditions. Of the latter, the most commonly used in hydrodynamics represent a stationary wall, as follows:

$$\begin{array}{ll} \text{frictionless}: & \mathbf{n} \cdot \dot{\mathbf{x}} = 0 \\ \text{adherent}: & \dot{\mathbf{x}}_{\mathsf{l}} = 0 \end{array} \tag{1.8-8}$$

To illustrate the interplay between these and Eulerian hydrodynamics, let us consider the classic rectilinear shearing, used again and again as an example of the effects of friction or the consequences of its absence:

$$\dot{x}_1 = 0, \ \dot{x}_2 = v(x_1), \ \dot{x}_3 = 0. \tag{4.3-12}$$

The plane $x_1 = \text{const.}$ is material and moves rigidly at the constant speed $v(x_1)$ in the direction of the x_2-axis. The condition (1.8-5) is satisfied on each of these planes. Thus we may think of this flow as representing an infinite body of fluid confined between the two parallel planes $x_1 = \pm d$, $d = \text{const.}$ The velocity field is isochoric and accelerationless: $\operatorname{div} \dot{\mathbf{x}} = 0$, $\ddot{\mathbf{x}} = 0$. If we suppose that $\mathbf{b} = \mathbf{0}$, the dynamical equation (9.1-3) is satisfied if and only if R is a function of t alone. For a homogeneous incompressible fluid, therefore, any constant pressure p suffices to maintain the flow (4.3-12) indefinitely, with any profile v. A rigid and hence irrotational transplacement in the direction of x_2 is one possibility. In Section 5.4 we saw that for incompressible viscous fluids of a very large class the profile v is determined by the dynamical equation and the boundary condition (1.8-8), which represents adherence of the fluid body to its bounding walls. That is not true of a body of Eulerian fluid, for it cannot support or effect shear traction upon any surface, exterior or interior. In the shearing flow presently under consideration, the planes $x_1 = \text{const.}$ may slide freely upon each other in any way, like a stack of smooth cards. Any profile v such that $v(\pm d) = 0$ satisfies (1.8-8). Thus the problem as set for an Eulerian fluid has many solutions, too many for useful interpretation.

On the other hand, we might try to determine a good solution by imposing a kinematical constraint, for example irrotational flow $\mathbf{W} = \mathbf{0}$. This condition makes the profile v in (4.3-12) a linear function, which cannot satisfy the condition (1.8-8) expressing adherence unless the flow reduces to a state of rest: $\dot{\mathbf{x}} = \mathbf{0}$ everywhere. Thus the irrotational solution, while unique, does not correspond with what we see in nature when a fluid body moves down a straight, deep canal. Such a flow of a real fluid is generally rotational, at least in portions near the confining walls; the spin varies with distance from the walls and is at least roughly determinate.

The simple, even trivial, class of flows we have just considered illustrates a rough, general idea: the Euler fluid is unsuited to describe the motions of natural fluids except, possibly, at places distant from confining walls. A deeper analysis of the concepts of Eulerian hydrodynamics is required. As a start, we turn again to general properties of Eulerian fluids.

First we take up some simple aspects of Euler's solution when it is applied to homogeneous incompressible fluids. From (2) and $(9.1\text{-}2)_2$ we read off the following theorem of Euler: *An isochoric velocity field provides a solution of Euler's dynamical equation for an incompressible fluid subject to lamellar body force if and only if its acceleration field is lamellar; its pressure field is then determined as follows*:

$$p v = P_{\mathfrak{a}} - \varpi. \tag{9.2-5}$$

This result can be expressed a little differently as *all flows of the kind specified are universal for homogeneous incompressible Eulerian fluids.*[2] The indeterminate function of time understood in (5) may be interpreted as the value of pv at all places where $P_{\mathfrak{a}} = \varpi$; often these places constitute surfaces that may be interpreted as boundaries, possibly at ∞.

For a compressible fluid the condition (2.2-29) is merely necessary, not sufficient, for a given velocity field to correspond to a solution of Euler's dynamical equation (9.1-1). The constitutive equation (4.1-8) prescribes the pressure as a function of ρ. The pressure function p is usually assumed to be such that $dp/d\rho > 0$. (This adscititious inequality ensures that weak waves may propagate throughout a body of the fluid; see Exercise 9-1 Then $(9.1\text{-}2)_1$ makes an increasing function of ρ, so (2) requires that

$$\rho = R^{-1}(P_{\mathfrak{a}} - \varpi). \tag{9.2-6}$$

On the other hand, (1.1-8) determines ρ from the given velocity field when ρ_κ is prescribed. Thus we have two different determinations of ρ. The given velocity field is a solution of Euler's dynamical equation if and only if suitable choices of ρ_κ and the constant understood in $P_{\mathfrak{a}} - \varpi$ make these two determinations

[2] See section 4.10 of C. Truesdell, *A First Course in Rational Continuum Mechanics.*

agree. "Universal solution" here would mean a flow that preserves circulation and renders the two determinations of ρ consistent for all invertible functions R. While it is known that certain homogeneous transplacements are universal when ϖ is suitably specialized, the problem of determining all universal flows of Eulerian compressible fluids seems not to have been studied as yet. A particular, very special family of universal solutions will be exhibited at the end of Section 3.

9.3 Accelerationless Flows: Hydrostatics

Directly from (9.1-1) we can read off *Clairaut's theorem: In a body of elastic fluid at rest the surfaces of constant pressure are normal to* **b**. The same holds under the weaker condition $\ddot{\mathbf{x}} = \mathbf{0}$; such a flow is called *accelerationless.*[3] The foregoing statements do not require the fluid body to be homogeneous.

For homogeneous bodies we may use (9.1-3) and so conclude that when $\ddot{\mathbf{x}} = \mathbf{0}$, then **b** is normal to the surfaces $R =$ const. By assuming that **b** has the potential ϖ, we derived (9.2-1); on the further assumption that $dp/d\rho > 0$, we derived (9.2-6). Thence we see that in an accelerationless flow, to within an arbitrary function of t,

$$p = -\rho\varpi \text{ for incompressible fluids} \tag{9.3-1}$$

and

$$\rho = T^{-1}(-\varpi),\ \ p = f(R^{-1}(-\varpi)) \text{ for compressible fluids such that } dp/d\rho > 0. \tag{9.3-2}$$

These statements include Euler's *general solution of the problem of hydrostatics.*

In Section 16 of Chapter 4 we remarked that *all fluids obey in permanent equilibrium the laws of Eulerian hydrostatics*. For those, of course, the conclusions (1) and (2) apply at any instant when $\ddot{\mathbf{x}} = \mathbf{0}$.

From (9.1-3) we see that $\ddot{\mathbf{x}} = \mathbf{b}$ if and only if R is a function of t alone. If we consider only compressible fluids such that $dp/d\rho > 0$, from (1.1-8) we see that R is a function of t alone if and only if J is. Consequently, for homogeneous fluid bodies subject to surface pressures alone, *an accelerationless flow is universal if and only if the corresponding field J is a function of t alone.*

[3] Accelerationless flows can be quite complicated. A kinematical analysis of them may be found on pages 100, 102, and 141 of C. Truesdell and R. Toupin, Classical Field Theory, *Handbuch der Physik* 3_1, (Berlin, Gottingen, and Heidelberg: Springer-Verlag, 1960).

9.4 Irrotational Flows: General Aspects

Irrotational flows were defined in Section 2.2 by the condition $\mathbf{W} = \mathbf{0}$.

Since the beginnings of mathematical hydrodynamics these flows have been studied intensively. Long before the kinematics of continua had been much developed, D'Alembert and Euler noticed that the formal statement $\mathbf{W} = \mathbf{0}$ in a region easily provides a solution of (9.2-1). Evolution of formal aspects preceded understanding of the concepts. In later times the status of irrotational flows as a special subclass of the flows satisfying Euler's dynamical equation (9.1-1) was clarified somewhat, though elements of uncertainty remain even now.

Before reading further, students would do well to review the pure kinematics of stretching and spin, presented above in Section 2.2. Recall the term "potential flow," which refers to the existence of a velocity potential P_{v} in a simply connected region:

$$\dot{\mathbf{x}} = -\operatorname{grad} P_{\mathrm{v}}. \tag{2.2-36}$$

Bear in mind the discussion at the end of the section, for they exclude potential flow other than a state of rest in many circumstances, roughly as follows:

1. An isochoric flow of a body of fluid that adheres to any stationary surface, however small.

2. A flow that is isochoric or has steady density, whose finite boundaries are stationary and for which $\rho P_{\mathrm{v}} \partial_r P_{\mathrm{v}}$ vanishes faster than $1/r^2$ at ∞.

We proceed now to develop mathematically some dynamic aspects of potential flows, postponing until Sections 6 and 7 our consideration of rotational flows of Eulerian fluids.

First, the velocity potential P_{v} determines the acceleration potential P_{a} as follows:

$$P_{\mathrm{a}} = P_{\mathrm{v}}' - (1/2)|\operatorname{grad} P_{\mathrm{v}}|^2. \tag{7.3-2}$$

Thus from (8.2-2) we learn that

$$R = P_{\mathrm{v}}' - (1/2)|\operatorname{grad} P_{\mathrm{v}}|^2 - \varpi. \tag{9.4-1}$$

The function R, defined by (9.1-2), is determined from the constitutive specification of the fluid and hence is a known function of ρ for a compressible fluid, while for a homogeneous incompressible fluid it is p/ρ. We think of ϖ as assigned. Thus (1), which expresses the Bernoullian theorem for potential flows, determines the pressure once P_{v} and ρ are known:

$$P_{\mathrm{v}}' - (1/2)|\operatorname{grad} P_{\mathrm{v}}|^2 - \varpi = \begin{cases} \int \frac{1}{\rho}\frac{dp}{d\rho}\, d\rho; \\ \frac{p}{\rho} \end{cases}$$

as usual, the function of time to within which ϖ is determined is not written, and the upper determination refers to compressible fluids, the lower, to incompressible ones. To obtain a second condition connecting P_v and ρ, we substitute $-\operatorname{grad} P_v$ for $\dot{\mathbf{x}}$ in $(1.5\text{-}6)_2$ and so obtain

$$\rho' - \operatorname{div}(\rho \ \operatorname{grad} \ P_v) = 0. \tag{9.4-2}$$

The simplest problems of hydrodynamics prescribe the initial value of grad P_v and ρ throughout a fixed region $\mathcal{V}$ and on $\partial\mathcal{V}$ assign the normal component of velocity. For irrotational flow, (1.8-1) becomes

$$\mathbf{n} \cdot \dot{\mathbf{x}} = -\frac{\partial P_v}{\partial n} = \text{a prescribed function of } \mathbf{x} \text{ and } t \text{ on } \partial\mathcal{V}. \tag{9.4-3}$$

If $\mathbf{b}$ is lamellar, the partial differential equations (1) and (3) provide a purely kinematical statement of Euler's dynamical equation (9.1-3) when irrotational flow is presumed. The two scalar variables P_v and ρ are to be determined by solving the system, which is of first order in t and of second order in $\mathbf{x}$. The boundary condition (4) is also kinematical. This reduction to *pure kinematics*, which because of its explicitly stated boundary condition (4) is plainer and more direct than the more general one we set forth in Section 2, could seem "to reduce the whole matter to pure geometry," but for compressible fluids great difficulties remain. Even good illustrative examples are rare.

It is otherwise if we descend to irrotational flows of homogeneous incompressible fluids. For them, as we saw in Section 2.2, the statement (3) reduces to

$$\Delta P_v = 0. \tag{2.2-37}$$

The problem of solving this partial differential equation subject to the boundary condition (4) is called the Neumann problem. A vast literature is devoted to it. When the Neumann problem has been solved for a particular boundary, the corresponding potential P_v may be put into the Bernoullian theorem (2), which then determines p/ρ to within an arbitrary function of t. Thus the *solution of the kinematical problem delivers at once the solution of the dynamical problem.*

9.5 Irrotational Flows: Reactions on Submerged Obstacles and Cavitites

In Section 2.2 we found conditions sufficient to make an irrotational flow past an obstacle impossible. We shall now calculate that reaction when the deforming body consists of homogeneous incompressible Eulerian fluid and undergoes irrotational flow.

Each Cartesian component of $\dot{\mathbf{x}}$ is a harmonic function, and a function harmonic in a region outside every sphere has a power-series expansion at ∞. Hence if $\dot{\mathbf{x}} \to \mathrm{v}$ as $r \to \infty$, it follows that

$$\dot{\mathbf{x}} = \mathrm{v} + \frac{1}{r}\mathbf{a} + \frac{1}{r^3}\mathbf{A}\mathbf{p} + \frac{1}{r^5}\hat{\mathbf{A}}(\mathbf{p}\otimes\mathbf{p}) + \mathbf{O}\left(\frac{1}{r^4}\right). \tag{9.5-1}$$

In this formula $\mathbf{p}$ denotes a position vector field, $r \equiv |\mathbf{p}|$, $\mathbf{a}$ is a constant vector field, $\mathbf{A}$ is a constant second-order tensor field, and $\hat{\mathbf{A}}$ is a constant third-order tensor field; $\hat{\mathbf{A}}$ and $\mathbf{A}$ are so chosen as to render each component of $\mathbf{Ap}$ and of $\hat{\mathbf{A}}(\mathbf{p}\otimes\mathbf{p})$ a spherical harmonic of first or second order, respectively. The expansion (1) is subject to the further requirement that $\operatorname{div}\dot{\mathbf{x}} = 0$, $\Delta\dot{\mathbf{x}} = \mathbf{0}$; as students will verify by a simple calculation, followed by passage to the limit as $r \to \infty$, this requirement entails the conditions $\mathbf{a} = \mathbf{0}$, $\mathbf{A} = A\mathbf{1}$. If we require also that the flux of mass through every sufficiently large sphere be null, it follows that $\mathbf{A} = \mathbf{0}$, so

$$\dot{\mathbf{x}} = \mathrm{v} + \frac{1}{r^5}\hat{\mathbf{A}}(\mathbf{p}\otimes\mathbf{p}) + \mathbf{O}\left(\frac{1}{r^4}\right),$$

$$-P_{\mathrm{v}} = \mathrm{v}\cdot\mathbf{p} + O\left(\frac{1}{r^2}\right). \tag{9.5-2}$$

We recall the condition (2.2-38), which would imply that $P_{\mathrm{v}} =$ const.; it is not satisfied unless $\mathrm{v} = \mathbf{0}$. In fact, it is easy to see that there are infinitely many nonconstant harmonic potentials P_{v} that represent nontrivial flows of the kind we are considering. Applying $(2)_2$ to (9.4-1) when $\varpi = 0$ and $P'_{\mathrm{v}} = 0$ shows that

$$\mathbf{p} = \text{const.} + \mathbf{O}\left(\frac{1}{r^3}\right). \tag{9.5-3}$$

And hence the "Euler-D'Alembert paradox" holds.[4]

THEOREM 9.1

Let an obstacle be submerged in a body of homogeneous incompressible elastic fluid undergoing steady irrotational flow. If the body force is null, and if no fluid is supplied or removed at ∞, the fluid exerts no force on the obstacle.

The student should note that the shape of the obstacle need not be connected. For example, it may be two disjoint spheres. In that case the fluid body generally exerts a non-null force on each sphere, but the sum of these forces must be null. None of the conditions stated as sufficient for the truth of the result is inessential. For example, in order to derive (3) it is necessary to assume that the flow is steady. Of course a fluid in unsteady flow generally exerts forces on an obstacle, as the

[4]See Section 2 of C. Truesdell, *A First Course in Rational Continuum Mechanics*, vol. 1, for a detailed discussion of reactions upon obstacles.

axioms of inertia suggest. Moreover, if the density of the fluid is not uniform, (9.4-1) generally does not hold, again (3) does not follow, and the proof fails. Supply or withdrawal of fluid at ∞ may result in forces, finite or infinite, upon submerged bodies. Finally, if the body force is not null, the obstacle and the fluid body generally exert non-null forces upon one another, as is illustrated by Archimedes' principle.

For irrotational flows of compressible fluids the "Euler-D'Alembert paradox" holds if the flow is everywhere subsonic: $|\dot{\mathbf{x}}|^2 < p'(\rho)$, but the proof is much more difficult[5] In a supersonic part of a flow, shock waves may occur, and then a submerged body generally experiences a force in the direction of $\mathbf{v}$. These matters go beyond the scope of this book.

Unless $\hat{\mathbf{A}}(\mathbf{p} \otimes \mathbf{p}) = \mathbf{0}$, the decay estimates of velocity, density, and pressure that ensure the boundedness of certain integrals do not hold. Consequently, while the fluid body in isochoric irrotational flow exerts no resultant force on an obstacle, it may exert upon the obstacle a torque, even an infinite torque.

There are various ways to outflank the Euler-D'Alembert paradox. One of these rests upon the simple fact that while in a 3-dimensional space a bounded obstacle may be enclosed in a sphere, the region exterior to which is simply connected, in a plane flow the exterior of a circle is doubly connected. A plane flow past an obstacle, necessarily plane, is interpreted as a three-dimensional flow past one or more infinite cylinders, the exterior of which is multiply connected. A plane isochoric irrotational flow around a two-dimensional obstacle corresponds, in general, to a cyclic potential. Those very considerations that in a three-dimensional space lead to (2) deliver for a plane flow the expansions

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{v} + \frac{C}{2\pi r^2}\mathbf{p}^{\perp} + \mathbf{O}\left(\frac{1}{r^2}\right), \\
-p &= \mathbf{v} \cdot \mathbf{p} + \text{const.} + \frac{C}{2\pi}\theta + O\left(\frac{1}{r^2}\right);
\end{aligned}
\tag{9.5-4}
$$

here the constant C is the circulation about any circuit embracing the origin once and described counterclockwise, $\mathbf{p}^{\perp}$ is a two-dimensional position vector rotated counterclockwise through a right angle, and θ is the azimuth of $\mathbf{p}$. The term whose coefficient is C in (4) corresponds to the *irrotational simple vortex* and C is circulation of that vortex. Thus the flow past the obstacle is obtained by superposing an irrotational vortex upon a flow with null circulation.

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[5] R. Finn and D. Gilbarg, "Three-dimensional subsonic flows, and asymptotic estimates for elliptic partial differential equations," *Acta Mathematica* 98 (1957): 265–296. L, Bers, "Existence and uniqueness of a subsonic flow past a given profile," *Comm. Pure and Applied Mathematics* 4 (1954): 441–504.

EXERCISE 9.5.1 (Kutta,[6,7,8] Joukowski[9,10])
Let an infinitely long cylindrical obstacle be immersed in a steady irrotational flow of a body of incompressible elastic fluid such that the velocity $\mathbf{v}$ at ∞ is perpendicular to the generators of the cylinder, and suppose that no fluid be supplied or withdrawn at ∞. Show that the force $\mathbf{f}_{obs}$ exerted upon a portion of that cylinder having the length L is given by

$$\mathbf{f}_{obs}/L = -\rho C \mathbf{V}^{\perp}. \tag{9.5-5}$$

The torque exerted by the fluid body upon a finite portion of the obstacle is finite and is independent of C.

The result of this exercise shows that a section of the submerged obstacle suffers no drag but experiences a definite lift, determined by the circulation C. In principle C may be assigned at will, so the problem remains indeterminate. If the contour has corners or cusps, thus generally are singularities at which the velocity does not exist; as they are approached, generally the speed becomes infinite. Joukowski gave reasons to impose the requirement that the speed should remain finite at such a singularity. Using the Riemann mapping theorem, one may show that if the contour is a simple closed curve with *a single sharp trailing edge that is not a cusp*, then there is exactly one circulation C in $(4)_1$ renders the given cylinder a boundary on which Joukowski's condition is satisfied.[11] Thus a determinate lift results. Certain conceptual difficulties remain. However closely the profile with the sharp trailing edge is approximated by a smooth one, the circulation and hence the lift remain arbitrary; a profile with two sharp edges does not have a determinate lift; and for a cylinder that is merely very long but not infinitely long, we cannot escape the conclusion of the three-dimensional theory of potential flow about a submerged obstacle. The fact remains that circulation, however it may be generated or explained, does give rise to a force on a submerged object, and, if that object is an infinitely long cylinder, the force per unit length is finite and perpendicular to the direction of the stream. This fact has been used to explain the "Magnus effect," on the basis of which the rotor ship, and the lift produced by the wing of an airplane was projected.

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[6] W.M. Kutta, "Auftriebskrafte in strömenden Flussigkeiten," *Illustrierte Aeronautische Mitteilungen* 6 (1902): 133–135; "Über eine mit den Grundlagen des Flugproblems in Beziehung stehende zweidimensionale Strömung," *Sitzungberichte der Bayerische Akademie der Wissenschaften, Mathematisch-Physikalische Klasse*, (1910), pp 1-58.

[7] W.M. Kutta, Sitzb cricut bay nistito Akad, eno Wissensinatie (Munich) 40. 1910.

[8] W.M. Kutta, "Über ebene Zirkulationsströmungen nebst flugtechnischen Anwendungen," *Sitzungsberichte der Bayerische Akademie der Wissenchaften, Mathematisch-Physikalische Klasse*, (1911): 65–128.

[9] N.E. Joukowski, "De la chute dans l'air de corps légers de forme allongee, animés d'un mouvement rotatoire," *Bulletin de l' Institut Aerodynamique de Koutchino*, 1 (1912): 51–66.

[10] N.E. Joukowski, "Über die Kontouren der Tragflachen der Drachanflieger," *Zeitschrift für Flugtechnik und Motorluftschiffahrt* 1 (1910): 281–84.

[11] In fact the velocity must vanish at the trailing edge.

The foregoing discussion presumes that the fluid body fills all of space except that occupied by the obstacle. If the fluid in passing the obstacle separates from it and streams on unhindered, the conclusions are different. The Euler-D'Alembert paradox does not reflect absence of pressures upon an object submerged in a stream but rather the equality of total front pressure and total back pressure. If a jet of fluid strikes a plate bluff onward, after having been deflected radially in all directions parallel to the plate the fluid mass generally will not close in again behind it but rather will stream on past, leaving a cavity. The plate, then being subject to pressure on its forward side only, will obviously suffer a drag. Flows of this kind can be idealized by supposing the cavity vacuous or filled with vapor or stagnant fluid so that the pressure on the free boundary is uniform. For a steady irrotational flow of a homogeneous incompressible fluid subject to no body force, we can appeal to $(9.4\text{-}2)_2$ and so conclude that the speed is constant along each free streamline. The partial differential equation (2.2-37) still governs the problem, and the boundary condition on the wetted part of the obstacle is still $\partial P/\partial n = 0$, but a solution is sought to satisfy this condition only as long as p remains greater than some given pressure, while the remaining boundary is determined by the conditions $|\operatorname{grad} P| = \text{const}$. Flows of this kind do exert forces and torques upon obstacles, and, in addition, the shapes of the free surfaces are of great interest. The mathematical theory is difficult.[12] Similar difficulties arise in the theory of surface waves. For this reason a large part of the vast literature of hydrodynamics is devoted to linearizations and other truncations in the interest of cutting down the problem to the range of the mathematical tools at the disposal of the investigator. The results of some of these simplified studies have been proved to have definite status as approximations.

EXERCISE 9.5.2 (Borda)
A jet of incompressible fluid in a steady irrotational flow issues from an infinite container into a semi-infinite straight tube of cross-sectional area A. Assume that $\dot{\mathbf{x}} = 0(r^{-2})$ as $\mathbf{x} \to \infty$ within the container, while as $\mathbf{x} \to \infty$ within the jet $\dot{\mathbf{x}} \to \mathbf{a}$, which is a constant vector parallel to the axis of the jet. As a first approximation, suppose that the pressure at the attachment of the tube to the container has the same value as that when the tube was not present, and show that the cross-sectional area at the vena contracta is $\frac{1}{2}A$.

Next, let $\partial\Sigma$ denote a cross-section of the container far above the opening of the tube. Let $\partial\varpi$ denote the boundary of the container below the lateral boundaries of the tube $\partial\Sigma$, $\partial\varpi$ and the jet cross-section $\partial\sigma$. Use Gauss' theorem, the facts

[12] See D. Gilbarg, "Jets and cavities," pp. 311–445 of *Encyclopedia of Physics* 9, edited by S. Flügge and C. Truesdell, (Berlin, Göttingen, and Heidelberg: Springer-Verlag), 1960; G. Birkhoff and E.H. Zarantonello, *Jets, Wakes and Cavities*, New York: Academic Press, (1957).

that the fluid is incompressible and that the flow is irrotational, to show that

$$\frac{A_j}{A} = \frac{1 - \frac{\mathbf{n}_\Sigma}{A} \int\limits_{\partial\varpi} \mathbf{n}\left(\frac{v^2}{v_j^2}\right) da}{2 - \frac{A_j}{A_\Sigma}\mathbf{n}_j \cdot \mathbf{n}_\Sigma}, \qquad (9.5\text{-}6)$$

where $\mathbf{n}$ is the unit outward normal, $\mathbf{n}_\Sigma$ and $\mathbf{n}_j$ being the outward normals to the boundaries $\partial\Sigma$ and the jet cross-section, both of which are assumed to be planar. A_j and v_j denote the cross-sectional area and the velocity of the jet, respectively.

If $\partial\Sigma$ is a plane whose normal is in a direction perpendicular to the axis of the jet, then $\mathbf{n}_\Sigma \cdot \mathbf{n}_j = 0$. Thus, for the flow through a hole in an infinite plane, we have $\frac{A_j}{A} = \frac{1}{2} + \frac{1}{2A} \int\limits_{\partial\varpi} \left(\frac{v^2}{v_j^2}\right) da > \frac{1}{2}$. Next, use (6) to show that when a vertical cylindrical tube is fitted to the horizontal bottom of a container, then $\frac{A_j}{A} > \frac{1}{2}$. However, when a horizontal cylindrical tube is fitted to a container so that the nozzle projects inwards, then the ratio of the cross-section A_j to A is nearly $\frac{1}{2}$. Such an apparatus is called Borda's mouthpiece.[13]

9.6 Rotational Flows: Vorticity, Bernoullian Theorems

This section concerns compressible elastic fluids as well as incompressible ones.

We might think we could evade the objectionable features of potential flows by resorting to rotational flows of elastic fluids. If we do so, we find that boundary-value problems that for irrotational flow have no solutions have possibly infinitely many solutions for rotational flows. For example, if we consider steady rectilinear flow of a homogeneous incompressible fluid sheared between two parallel planes, we could refer to (5.4-1), which expresses the principle of balance of linear momentum, reduces to $p = -\rho\varpi + \text{const.}$ and imposes no restriction upon the velocity profile v. Alternatively, we can see from (5.4-5) that $a = 0$ and $c = 0$, so (5.4-9) reduces to $0 = 0$ and thus is satisfied by any $\mathbf{v}$. If the velocities of the plates $x_1 = a$ and $x_2 = b$ are $\mathbf{v}_a$ and $\mathbf{v}_b$, any function $\mathbf{v}$ such that $\mathbf{v}(a) = \mathbf{v}_a$ and $\mathbf{v}(b) = \mathbf{v}_b$ solves the problem for the incompressible elastic fluid. Unless $\mathbf{v}_a = \mathbf{v}_b$, all these flows are rotational.

These conclusions are obvious from the constitutive equations (4.1-8) and (4.1-9). An elastic fluid cannot exert shearing stress. Therefore it slides along walls without dragging them. Nor can a plane wall impede or promote the tangential

[13] A discussion of a two dimensional form of the mouthpiece using conformal mapping techniques can be found in L.M. Milne-Thomson, *Theoretical Hydrodynamics* (London: Macmillan, 1949).

flow of an elastic fluid. Thus we cannot expect the elastic fluid to provide any information about the "frictional drag" exerted by real fluids on real walls. This drag is associated with viscosity, which is the subject of study in Chapters 5–7.

Rotational flows of elastic fluids, compressible or incompressible, are of interest in themselves. Their great variety, which leads to indeterminacy, lessens the value of particular cases, but their general nature can be grasped through several glorious theorems discovered by the great hydrodynamicists of the nineteenth century. As we stated and proved generalizations of those theorems in kinematical terms in Chapter 2, here we need only bring together their applications of elastic fluids subject to lamellar body force. The D'Alembert-Euler condition (9.2-2) then holds. Some of the consequent conclusions we may phrase in colloquial terms as follows:

1. The circulation of every material circuit in a simply connected region is preserved (Kelvin's theorem, sometimes called the *fundamental theorem of classical hydrodynamics*).

2. At a given time, the flux of vorticity through each and every cross-section of a vortex tube is the same (Helmholtz's first theorem, an immediate consequence of the fact that the vorticity field is solenoidal).

3. A material line once a vortex line is always a vortex line (Helmholtz's second theorem).

4. The flux of vorticity through each material surface is constant in time (Helmholtz's third theorem).

5. For each finite material curve C that is a vortex line,

$$\int_C \frac{ds}{JW} = \text{const. in time} \qquad (9.5\text{-}7)$$

(Appell's theorem).

6. Cauchy's vorticity formula:

$$\mathbf{F}^T\mathbf{W}\mathbf{F} = \mathbf{W}_\kappa. \qquad (2.2\text{-}26)$$

Thus vorticity is only transported, never created or destroyed. A corresponding differential statement is the D'Alembert-Euler condition:

$$\dot{\mathbf{W}} + \mathbf{D}\mathbf{W} + \mathbf{W}\mathbf{D} = \mathbf{0}. \qquad (9.2\text{-}1)$$

7. If a velocity potential exists on the shape of a body of elastic fluid at one time, it exists on the shape of that body at every other time (Cauchy's velocity-potential theorem).

EXERCISE 9.6.1 (Hankel, Kirchoff)
Helmholtz's vorticity theorems follow at once from Cauchy's vorticity formula.

EXERCISE 9.6.2 (Hankel, Appell)
(2.2-25) and (2.2-29) may be expressed as follows

$$\operatorname{skw} \operatorname{Grad}(\mathbf{F}\ddot{\mathbf{x}}) = \mathbf{0}, \tag{9.6-1}$$

$$\mathbf{F}\ddot{x} = -\operatorname{Grad} P_{\mathfrak{a}}. \tag{9.6-2}$$

The notation used in (1), traditional in hydrodynamics, denotes by the symbols introduced for certain functions of the spatial variables the results of transforming those functions to referential variables; for example, the $P_{\mathfrak{a}}$ in (2) would be denoted by $P_{\mathfrak{a}} \circ \chi_{\kappa}$ in a modern notation.

The foregoing theorems make rotational flows "approachable in concept," as Helmholtz said. The spin has a certain permanence, and sometimes (2.2-24) is said to describe the "convection of vorticity." Indeed, *the present local deformation alone determines the present spin* from the spin of the same fluid point at any other time, *regardless of the motion of that point at intermediate times*. These theorems have been much used to explain phenomena in real fluids in regions far from solid boundaries.

There is a simple Bernoullian theorem for rotational flows, provided the spin field is steady: $\mathbf{W}' = \mathbf{0}$, because then we may use the purely kinematical theorem

$$\ddot{\mathbf{x}} = \operatorname{grad}\left(Q + \frac{1}{2}|\dot{\mathbf{x}}|^2\right) + \mathbf{W}\dot{\mathbf{x}}, \tag{7.4-3}$$

where Q is a scalar field. For lamellar body force with potential ϖ, Euler's dynamical equation (9.1-3) takes the form

$$\mathbf{W}\dot{\mathbf{x}} = -\operatorname{grad}(R + \varpi + Q + (1/2)\,|\dot{\mathbf{x}}|^2), \tag{9.6-3}$$

when R is defined by (9.1-2). If $\mathbf{W}\dot{\mathbf{x}} = \mathbf{0}$, (3) yields

$$R + \varpi + Q + (1/2)\,|\dot{\mathbf{x}}|^2 = k, \tag{9.6-4}$$

in which k is a function of time alone. If $\mathbf{W} = \mathbf{0}$, (4) reduces to (9.4-1), for then $Q = -\mathrm{P}'$. If $\mathbf{W} \neq \mathbf{0}$, the condition $\mathbf{W}\dot{x} = \mathbf{0}$ requires the axis of spin to contain the velocity vector, so the fluid points may be said to spin about their paths; a flow in which this condition is satisfied is called a *screw flow*, and (4) expresses a Bernoullian theorem for motions of this kind. Finally we consider the typical case, in which $\mathbf{W}\dot{\mathbf{x}} \neq \mathbf{0}$. Then the axis of spin and the direction of the velocity span a plane at each point. This plane is the *Lamb plane*, and $\mathbf{W}\dot{\mathbf{x}}$ is normal to it. According to (3), the field of Lamb vectors has a normal congruence of surfaces; equivalently, the Lamb planes are the tangent planes of these surfaces, which are called *Lamb surfaces*. A Lamb surface may be regarded as swept out by streamlines and vortex lines. Interpretation of (3) now yields Lamb's *superficial Bernoullian theorem: If a flow of an elastic fluid, whether compressible or incompressible, is subject to lamellar body force and has a steady spin field and if it is neither an irrotational flow nor a screw flow, it has a congruence of Lamb surfaces $\mathcal{L}$. Moreover on each $\mathcal{L}$ there is a function $k_{\mathcal{L}}$ of time alone such that*

$$R + \varpi + Q - (1/2)\,|\dot{\mathbf{x}}|^2 = k_{\mathcal{L}}. \tag{9.6-5}$$

Since the vortex lines are steady, the motion of the Lamb surfaces is of a restricted kind. For example, in a plane flow the Lamb surfaces are cylinders perpendicular to the plane of flow and tangent to the velocity field. If the flow is not steady, the cross-sections of those cylinders are generally bent and stretched as the motion proceeds. Of course, if the flow is steady, so are the Lamb surfaces, and we may take Q as zero. For incompressible fluids, (5) implies that a kinematically determined curve is replaced by $\mathcal{L}$, which denotes a kinematically determined surface. In particular, in a steady flow

$$\varphi + 1/2\,|\dot{\mathbf{x}}|^2 = \text{const.} \tag{9.6-6}$$

in each streamline and each vortex line on $\mathcal{L}$, and the constant is the same for all those streamlines and vortex lines.

9.7 Rotational Flows: Gerstner Waves

A beautiful plane rotational flow discovered by Gerstner has become classic. Components of the transplacement that defines it are given as follows in terms of two parameters α and β:

$$\begin{aligned} x &= f(\alpha, \beta, t) := \alpha + \frac{1}{k} e^{-k\beta} \sin k(\alpha + ct), \\ z &= g(\alpha, \beta, t) := \beta + \frac{1}{k} e^{-k\beta} \cos k(\alpha + ct), \end{aligned} \tag{9.7-1}$$

where k and c are positive constants. We interpret x as a horizontal distance and z as a vertical distance taken positive downward. We note that $z \to \beta$ as $k\beta \to \infty$; thus large values of $k\beta$ correspond with great depths. The coordinates of the initial position $\mathbf{x}$ of a fluid point are given as follows in terms of the variables introduced in writing (1):

$$\begin{aligned} X &= f(\alpha, \beta, 0) = \alpha + \frac{1}{k} e^{-k\beta} \sin k\alpha, \\ Z &= g(\alpha, \beta, 0) = \beta + \frac{1}{k} e^{-k\beta} \cos k\alpha. \end{aligned} \tag{9.7-2}$$

EXERCISE 9.7.1
Verify that

$$\frac{\partial(x, z)}{\partial(\alpha, \beta)} = \frac{\partial(X, Z)}{\partial(\alpha, \beta)} = 1 - e^{-2k\beta}. \tag{9.7-3}$$

We shall suppose henceforth that $\beta > 0$. Then (3) shows that the motion is isochoric and that both (1) and (2) are locally invertible; in fact they are invertible over the whole range of X and Z consistent with the condition $\beta > 0$. Thus we may regard α and β as specifying points in a reference placement. The relations (1) with α and β interpreted as referential variables serves as an example of an isochoric motion specified in terms of a reference placement that is not an isochoric image of any placement occupied by the body in the course of its motion. The referential variables α and β are more convenient for the analysis than would be X and Z, as will be shown.

The transplacement (1) represents an undulation. If we replace α by $\alpha + 2\pi/k$, we increase x by $2\pi/k$ and leave z unchanged. Thus, if

$$\lambda \equiv \frac{2\pi}{k}, \tag{9.7-4}$$

we may call λ the wavelength of the disturbance. A fluid point on the material surface $\beta =$ const. occupies places with the same depth at all times such that $\alpha + ct =$ const. and hence such that $x + ct =$ const. Thus the shape of the surface $\beta =$ const. at all times may be obtained by causing its shape at some one time to move in the direction of decreasing x at the speed c. Therefore, the transplacement (1) represents a train of parallel, *periodic surface waves* of wavelength λ, which *propagate* unchanged in form at the speed c upon an infinitely deep and broad basin. The *frequency* of the waves is c/λ. If we agree to mean by the *amplitude a* of the vibration at the surface $\beta =$ const. one half the height of the crest above the trough, then

$$a = \frac{1}{k} e^{-k\beta}, \tag{9.7-5}$$

and

$$\frac{2\pi a}{\lambda} = e^{-k\beta}. \tag{9.7-6}$$

The path of the fluid point at (X, Z) initially is a circle about α, β having the radius a; the linear speed of the fluid point is $ce^{-k\beta}$. In these statements the values of α and β are determined from X and Z through (2). The whole train of waves is the result of rotations of the fluid points in circular orbits, each counterclockwise and at constant speed. Because $Z \to \beta$ as $k\beta \to \infty$, at great depths the amplitude of the waves becomes arbitrarily small. Thus (1) represents surface waves traveling straight across a basin of infinite extent and depth.

EXERCISE 9.7.2 (Gerstner)
Show that the profile of the surface $\beta = \beta_0 > 0$ is a trochoid obtained by rolling a circle of radius $1/k$ on the underside of the line $Z = \beta_0 - 1/k$, the distance of the tracing point from the center being $e^{-k\beta_0}/k$. If $\beta = 0$ in (1), the resulting shape V_{β_0} is a cycloid, and negative values of β are excluded because they would make two distinct fluid points occupy the same place. The maximum amplitude A of Gerstner's waves is bounded as follows:

$$A < \frac{\lambda}{2\pi}. \tag{9.7-7}$$

EXERCISE 9.7.3
Show that the vorticity, taken as positive when it corresponds to a positive clockwise circulation, is given by

$$w = \frac{2kce^{-2k\beta}}{1 - e^{-2k\beta}}; \tag{9.7-8}$$

also

$$W = e^{-k\beta} = \frac{2\pi a}{\lambda} < 1. \tag{9.7-9}$$

From (8) we see that Gerstner's waves are rotational, that their spin is the same at all points on the surface β = const., and that *the sense of the spin of the flow is the opposite of the sense of rotation of the fluid points in their orbits.* Also the rotationality of the flow decreases with $k\beta$; and equivalently with the ratio a/λ.

Next, using the notations $\partial_\alpha x := \partial_\alpha f(\alpha, \beta, t)$, $\partial_\alpha X := \partial_\alpha f(\alpha, \beta, 0)$, and so on we find that

$$\begin{aligned} \ddot{x}\partial_\alpha x + \ddot{z}\partial_\alpha z &= -kc^2 e^{-k\beta} \sin k(\alpha + ct), \\ \ddot{x}\partial_\beta x + \ddot{z}\partial_\beta z &= kc^2 e^{-k\beta} \cos k(\alpha + ct) + Kc^2 e^{-2k\beta}. \end{aligned} \tag{9.7-10}$$

It is easily seen by inspection that (9.6-1) and (9.6-2) hold for all reference placements if they hold for any. From (10) it is easy to see that (9.6-1) holds and to conclude that

$$-P_a = c^2 e^{-k\beta} \cos k(\alpha + ct) - 1/2c^2 e^{-2k\beta}. \tag{9.7-11}$$

Thus Gerstner's waves preserve circulation and hence solve Euler's dynamical equation if the body force is lamellar.

To consider the effect of gravity, we take ϖ as $-gz$, g being the gravitational acceleration, assumed constant. Then Euler's general solution shows that (4.8-15)

$$\begin{aligned} pv &= gz + P_a \\ &= g[\beta + \frac{1}{k} e^{-k\beta} \cos k(\beta + ct)] - c^2 e^{-k\beta} \cos k(\alpha + ct) + 1/c^2 e^{-2k\beta}. \end{aligned} \tag{9.7-12}$$

As usual, an arbitrary function of time is understood on the right-hand side. To make the solution represent waves upon a free surface, we require that $p =$ const. there. For the material surfaces $\beta =$ const. to be surfaces of constant pressure, it is necessary and sufficient to relate as follows the constants k and c:

$$c^2 = \frac{g}{k}. \tag{9.7-13}$$

From (4) we see that

$$2\pi c^2 = \lambda g, \tag{9.7-14}$$

(12) reduces to

$$\begin{aligned} pv &= g\left(\beta + \frac{1}{2k}e^{-2k\beta}\right) + \text{const.} \\ &= g\beta + 1/2\,\dot{x}^2 + \text{const.} \end{aligned} \tag{9.7-15}$$

Thus the wave length λ, which is an arbitrary constant, determines the speed of propagation. The pressure is determined to within assignment of an arbitrary value of the topmost surface.

Although $ce^{-k\beta}$ is the magnitude of the velocity at points on the surface $\beta =$ const., we must resist the temptation to regard (15) as "Bernoulli's theorem"; if the vorticity were steady, which it is not, (8.6-5) would hold, but there the term $(1/2)\dot{x}^2$ would have the opposite sign, and $g\beta$ is not the potential energy due to gravity. Certainly, however, (15) is an equation of Bernoullian type in that it determines p explicitly from the known velocity field.

To consider the behavior of Gerstner's wave at great depth, we let x tend to ∞ while the arbitrarily assigned wavelength λ is held fixed. Then from (1) and (2) we see that $(z - Z) \to 0$, $(x - X) \to 0$; and from (5) and (8) that $a \to 0$ and w $\to 0$. The relation (14), which determines the speed of propagation c from an arbitrarily assigned wavelength λ, is unaffected. Thus, if we consider Gerstner's solution only for values of β greater than some large constant, it represents *irrotational surface waves* of arbitrary wavelength and very small amplitude upon an infinitely deep and broad basin.

The statement (14), which was first inferred by Newton but with an incorrect numerical coefficient, is celebrated; Lagrange obtained a wholly wrong substitute for it, and, as far as we can learn, it was first obtained correctly in the way we have developed here not through the common and facile device of linearizing the problem itself but through a solution of it. The solution of this problem is a classic achievement of Levi-Civita[14] According to his conclusion, the ratio $2\pi c^2/(\lambda g)$ depends upon the amplitude of the waves.

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[14] A short proof is given by H. Berckert, "Existenzbeweise in der Theorie permanenter Schwerewellen einer inkompressiblen Flussigkeit langs eines Kanals," *Archive for Rational Mechanics and Analysis* 9 (1962): 379–394.

In Gerstner's solution the pressure remains constant at each fluid point. According to a theorem of Burnside, no nonconstant plane irrotational flow has this property. Thus a single irrotational solution, unlike Gerstner's, cannot be cut off at different depths to provide infinitely many solutions having different amplitudes at the free surface.[15]

9.8 Stokes' Conjecture for the Height of Irrotational Waves

In his celebrated paper on the motion of irrotational waves, Stokes'[16] inquired into the shape of the wave of greatest height in an Euler fluid, taking into consideration the effects of gravity but ignoring those of surface tension. On the basis of his calculations he conjectured that the boundary of the crest loses its differentiability and is made up of two convex functions whose tangents at the apex make an angle of $\frac{2\pi}{3}$. The conjecture was finally verified by Amick, Fraenkel, and Toland, [17], more than a century later. Some knowledge of functional analysis is required to follow the arguments that led to the resolution of Stokes' conjecture, students unfamiliar with functional analysis will find the requisite material in the Appendix.

In order to formulate the mathematical problem, we shall start by defining the domain in which the flow takes place. We shall assume that the flow is steady and plane and that $x_2 = Y(x_1)$ defines the location of the free surface of the body of fluid. Let

$$\Omega := \{(x_1, x_2) | -\infty < x_1 < \infty, -\infty < x_2 < Y(x_1)\} \tag{9.8-1}$$

denote the domain in which the flow takes place, and let the free surface Γ be defined through

$$\Gamma := \{(x_1, Y(x_1)) | -\infty < x_1 < \infty\}. \tag{9.8-2}$$

We shall assume that $Y(x_1)$ is periodic in x_1 with period $\lambda > 0$ and that the free surface Γ has only one crest per wave length. We shall also assume that the free

[15] The little exact knowledge of surface waves there was up to 1960 is summarized in Chapter F of the article by J.V. Wehausen and E.F. Laitone, "Surface Waves," *Handbuch der Physics* vol. 9, edited by S. Flügge and C. Truesdell, (Berlin, Gottingen, and Heidelberg: Springer-Verlag, 1963), 446–778.

[16] G.G. Stokes, "Considerations relative to the greatest height of oscillatory waves which can be propagated without change of form," *Mathematical and Physical Papers*, vol. 1 (Cambridge University, 1880), 225–28.

[17] C.J. Amick, L.E. Fraenkel, and J.F. Toland, "On the Stokes conjecture for the wave of extreme form," *Acta Mathematica* 148 (1982): 193–214.

surface Γ is symmetric around the crest of the wave. Then

$$H := \max_{-\infty<x_1<\infty} Y(x_1) - \min_{-\infty<x_1<\infty} Y(x_1) \tag{9.8-3}$$

is a measure of the amplitude of the wave. Stokes made several formal calculations under the assumption that (H/λ) is small in his quest to determine the form of the wave of greatest height.

Since the flow is planar, we can introduce the stream function q through

$$\mathbf{u} = (\nabla q)^{\perp}, \tag{6.3-2}$$

and, as the free surface is periodic, we shall seek a solution in which the stream function is also assumed to be periodic, *i.e.*,

$$q(x_1 + \lambda, x_2) = q(x_1, x_2). \tag{9.8-4}$$

Because the free surface is a streamline, we shall set

$$q(x_1, x_2) = 0 \text{ on } \Gamma, \tag{9.8-5}$$

and it also immediately follows that

$$\frac{1}{2}|\nabla q|^2 + gx_2 = \text{const on } \Gamma, \tag{9.8-6}$$

where g is the acceleration due to gravity. In order to complete the formulation of the problem, we shall have to specify the conditions as $x_2 \to -\infty$. To do this, let us introduce a coordinate system that is moving with the crest of the wave whose speed is $-c$ along the x_1 direction with respect to the quiescent fluid at infinity. Then, with respect to this moving frame,

$$\left(\frac{\partial q}{\partial x_1}, -\frac{\partial q}{\partial x_2}\right) \to (c, 0) \text{ as } x_2 \to \infty. \tag{9.8-7}$$

The Stokes conjecture was resolved by converting the problem defined by the equations of motion for an Euler fluid and (4)–(7) into an equivalent integral equation and by studying the properties of the integral operator that is usually called Nekrasov's integral.

We now proceed to outline the method for obtaining the equivalent integral equation from (4)–(7) and the equations of motion for an Euler fluid.

Let

$$\mathcal{D} := \left\{(x_1, x_2) \mid -\frac{\lambda}{2} \leq x_1 \leq \frac{\lambda}{2}, -\infty < x_2 < Y(x_1)\right\}. \tag{9.8-8}$$

Then $\mathcal{D}$ denotes the region occupied by a single wave. Let us first map $\mathcal{D}$ conformally onto a unit disk Δ centered at the origin. Let A and C stand for the troughs and B stand for the crest of the wave. Let z denote the complex number

$$z = x_1 + \mathrm{i}x_2, \tag{9.8-9}$$

where $i = \sqrt{-1}$ and ξ is the image of z under the mapping given by

$$\xi = \alpha_1 + i\alpha_2. \tag{9.8-10}$$

The part of the free surface $\{(x_1, Y(x_1)) \mid -\frac{\lambda}{2} \le x_1 \le \frac{\lambda}{2}\}$ is mapped onto the perimeter of the disk Δ, while the boundary at $x_2 \to -\infty$ is mapped into the center of the disk Δ. The part of the imaginary axis given by $\{(0, x_2) \mid -\infty < x_2 \le B\}$ is mapped into the part of the real axis, $\{(\alpha_1, 0) \mid 0 < \alpha_1 \le 1\}$. It is necessary to introduce a branch cut, $\{(\alpha_1, 0) \mid -1 < \alpha_1 < 0\}$, to accommodate the images of the line segments, $\{(-\frac{\lambda}{2}, x_2) \mid -\infty < x_2 \le A\}$ and $\{(\frac{\lambda}{2}, x_2) \mid -\infty < x_2 \le B\}$. It should be clear from the description of the conformal mapping that it should have a pole $\xi = 0$ and a branch cut on the negative real axis. With this in mind we seek a conformal mapping of the form

$$\frac{dz}{d\xi} = k\left(\frac{1}{\xi} + a_1 + a_2\xi + a_3\xi^2 + \dots\right). \tag{9.8-11}$$

We choose the constant $k = \frac{i\lambda}{2\pi}$ so that α_1 passes through 2π radians around the perimeter of Δ, then x_1 transverses one wavelength λ. It follows from (11) that

$$z = \frac{i\lambda}{2\pi}(\log\xi + a_1\xi + \dots), \tag{9.8-12}$$

and a simple calculation shows that all the a_i's have to be real if the wave is to be symmetric about the x_2-axis.

Let

$$\Theta(\xi) := \frac{\xi}{k}\frac{dz}{d\xi} := R(\xi)e^{i\theta(\xi)} \text{ on the surface.} \tag{9.8-13}$$

We shall see later that $\theta(\xi)$ denotes the angle that the velocity $\mathbf{u}$ of material points on the free surface makes with the x_1-axis. If the free surface Γ is parameterized by $\{\xi = e^{i\gamma} \mid -\pi \le \gamma \le \pi\}$ in the ξ-plane, then

$$\Theta(\gamma) = R(\gamma)e^{i\theta(\gamma)}. \tag{9.8-14}$$

EXERCISE 9.8.1

Using the Taylor series for the logarithm function, show that θ and R are related through

$$\theta(\xi) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{d}{dt}\log R(t)\sum_{n-1}^{\infty}\frac{\sin nt\sin\xi}{n}dt. \tag{9.8-15}$$

We shall show later that this integral for $\theta(\xi)$ leads to the integral equation of Nekrasov.

The boundary conditions (5)–(7) on the stream function reduce to

$$q = 0 \text{ on the unit disk,}$$
$$q = \infty \text{ at the center of the disk,}$$

$$\frac{\partial q}{\partial x_1} = 0, \frac{\partial q}{\partial x_2} = -c \text{ at the center of the disk.} \tag{9.8-16}$$

It can be shown that on the surface of the wave

$$\dot{x}_1 + i\dot{x}_2 = \frac{-c}{\Theta(\gamma)} = \frac{-ce^{-i\theta(\gamma)}}{R(\gamma)}, \tag{9.8-17}$$

and thus $(c/R(\gamma))$ denotes the magnitude of the velocity and $\theta(\gamma)$ describes the angle the velocity vector makes with the x_1-axis. Substituting (7) into (6), we obtain

$$\frac{c^2}{R^2(\gamma)} + 2g\hat{x}_2(\gamma) = \text{const.}, \tag{9.8-18}$$

where $\hat{z}(\gamma) = \hat{x}_1(\gamma) + \hat{x}_2(\gamma)$, with Γ being parameterized by γ. On differentiating (18) with respect to γ, we obtain

$$\frac{c^2}{R^3(\gamma)}\frac{dR}{d\gamma} = g\frac{d\hat{x}_2}{d\gamma}. \tag{9.8-19}$$

When $\xi = e^{i\theta(\gamma)}$, the chain rule yields

$$\frac{d\hat{z}}{d\gamma} = \frac{d\hat{z}}{d\xi}\frac{d\xi}{d\gamma} = -\frac{\lambda}{2\pi}\Theta(\gamma) = -\frac{\lambda}{2\pi}R(\gamma)e^{i\theta(\gamma)}, \tag{9.8-20}$$

where we have used (13). Thus,

$$\frac{d\hat{x}_2}{d\gamma} = -\frac{\lambda}{2\pi}R(\gamma)\sin\theta(\gamma). \tag{9.8-21}$$

It immediately follows from (20) and (21) that

$$\frac{c^2}{R^3(\gamma)}\frac{dR}{d\gamma} = \frac{\lambda g}{2\pi}R(\gamma)\sin\theta(\gamma), \tag{9.8-22}$$

which can be integrated to yield

$$R^{-3}(\gamma) = C\left[D + \int_0^{\gamma} \sin\theta(\tau)d\tau\right], \tag{9.8-23}$$

where D is a constant of integration and $C = \frac{3\lambda g}{2\pi c^2}$ is also a constant. Taking the logarithm of both sides of (23), differentiating with respect to γ and using (15), we obtain

$$\theta(s) = \frac{1}{3}\int_0^{\pi} K(s,t)\frac{\sin\theta(t)}{[D + \int_0^t \sin\theta(t)dr]}\,dt,\ 0 < s \le \pi, \tag{9.8-24}$$

where

$$K(s,t) = \frac{1}{\pi}\log\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)} = \sum_{k=1}^{\infty}\frac{\sin kt \sin ks}{k}. \tag{9.8-25}$$

Integral (24) with the kernel given by (25) is called Nekrasov's integral. Nekrasov[18] studied solutions to the integral that represented waves of small amplitude. The first investigator to study solutions to Nekrasov's integral equation that correspond to waves of finite amplitude was Krasovskii. He proved that for each value $0 < \alpha < \frac{\pi}{6}$, where

$$\alpha := \sup_{s\in[0,\pi]} \theta(s),$$

there is a finite value of D such that there is a continuous solution $\theta(s)$ to the Nekrasov integral. Krasovskii[19] conjectured that if the class of wave forms are restricted to those with smooth differentiable boundaries (Stokes' conjecture concerns nondifferentiable boundaries), then the angle between the surface of the wave and the horizontal exceeds $\frac{\pi}{6}$ for all finite values of D. This conjecture was proved to be false by McLeod.[20] The proof is quite involved and we shall not discuss it here.

Within the context of the integral equation (24), Stokes' conjecture reduces to the existence of a solution $\theta(s)$ such that

$$\lim_{s\to 0} \theta(s) = \lim_{s\to 0} \theta(s) = \frac{\pi}{6}, \tag{9.8-26}$$

where $D = 0$. Toland[21] proved that there is a solution $\theta(s)$ when $D = 0$ that is continuous in the interval $[-\pi, \pi]$ except at the point $s = 0$, where it suffers a discontinuity. While the nature of the discontinuity was not determined, it was shown that (9.8-26) would hold if the discontinuity were a jump discontinuity. Amick, Fraenkel, and Toland proved that the only possible discontinuity for $\theta(s)$ is the jump discontinuity, thereby verifying Stokes' conjecture.

We now proceed to sketch the existence proof. The interested student should read the relevant papers that have been cited for details regarding the proof.

The idea behind the proof is to obtain a solution $\theta_D(s)$, when $D \neq 0$, and then show that $\theta(s) = \lim_{D\to 0} \theta_D(s)$ is a solution. Keady and Norbury[22] proved that there are solutions to Nekrasov's equation when $D \neq 0$.

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[18] A.I. Nekrasov, "The exact theory of steady-state waves on the surface of a heavy liquid," Technical Summary Report No. 813, Mathematics Research Center, University of Wisconsin 1967 (D.V. Thapuran, ed. C.W. Cryer).

[19] Yu.P. Krasovskii, "On the theory of steady waves of finite amplitude," *U.S.S.R. Computational Mathematics and Mathematical Physics* 1 (1962): 996–1018.

[20] J.B.McLeod, "The Stokes and Krasovskii conjectures for the wave of greatest height," Technical Summary Report No. 2041, Mathematics Research Center, University of Wisconsin, 1980.

[21] J.F. Toland, "On the existence of a wave of greatest height and Stokes' conjecture," *Philosophical Transactions of the Royal Society of London*, Series A 363 (1978) 469–485.

[22] G. Keady and J. Norbury, "On the existence theory for irrotational water waves," *Transactions of the Royal Society*, Series A 83 (1978), 137–157.

THEOREM 9.2

For $D > 0$, but sufficiently small, there is a solution $\theta_D(s)$ of (24) that is continuous on $[-\pi, \pi]$, is odd, with $\theta_D(s) > 0$ when $s \in (0, \pi)$, $\theta_D(\pi) = 0$, and $|\theta_D(s)| < \frac{\pi}{2}$ when $s \in [-\pi, \pi]$.

Since this sequence of functions θ_D is continuous and bounded pointwise by $\frac{\pi}{2}$, the sequence is also bounded in $L^2[-\pi, \pi]$. Similarly, the sequence $\sin(\theta_D)$ is bounded in the same space. We can then extract a subsequence of θ_D, again denoted by θ_D, so that

$$\begin{aligned} \theta_D &\to \theta \quad \text{weakly in } L^2, \\ \sin(\theta_D) &\to \gamma \quad \text{weakly in } L^2. \end{aligned} \tag{9.8-27}$$

First, we need to prove that θ is not the trivial solution. Second, the function γ must be the sine of θ. Finally, θ must satisfy (24) with $D = 0$. These issues will be resolved by a series of lemmas. We begin by stating a technical lemma.

LEMMA 9.3

Let θ_D be any solution of (24). If D is sufficiently small and $0 \le \theta_D(s) < \pi/2$ for all $s \in [0, \pi]$, then there is a constant $\beta > 0$ such that for all $s \in [0, 2\pi]$

$$\theta_D(s) \ge \beta, \tag{9.8-28}$$

and

$$D + \int_0^t \sin\theta_v(s)ds \ge \frac{2}{3\pi}. \tag{9.8-29}$$

The proof of this lemma appears on page 474 of the article by Toland and will not be reproduced here. The two estimates (28) and (29) show that the solution θ_D is not the trivial function. The next lemma shows that this property is shared by the weak limit θ in (27) as well.

LEMMA 9.4

The function θ in (27) satisfies

$$\int_0^\pi \theta(s)ds > 0, \tag{9.8-30}$$

so that θ is not the trivial function.

PROOF We have two sequences θ_D and $\sin\theta_D$ that have weak limits θ and γ as $D \to 0$. If we let $g \equiv 1$ (A.6), then

$$D + \int_0^\pi \sin\theta_D(s)ds \to \int_0^\pi \gamma(s)ds, \tag{9.8-31}$$

but (29) implies that

$$\int_0^\pi \gamma(s)ds \geq \frac{2}{3\pi}. \tag{9.8-32}$$

On the other hand, the sequence θ_D converges weakly to θ so that (again taking $g = 1$ in (A.6))

$$\int_0^\pi \theta(s)ds = \lim_{\upsilon \to 0} \int_0^\pi \sin\theta_D(s)ds \int_0^\pi \lambda(s)ds \geq \frac{2}{3\pi}, \tag{9.8-33}$$

where we have used the fact that on the interval $[0, \pi]$ the inequality $\sin x \leq x$ holds. This completes the proof of the lemma.

The next lemma gives a very useful reinterpretation of the Nekrasov integral equation.

LEMMA 9.5

Let the function ψ be defined by

$$\psi(t) = \log\left[D + \int_0^t \sin\theta_D(s)ds\right]. \tag{9.8-34}$$

Let $C\psi$ stand for the conjugate of ψ (see the Appendix). Then the Nekrasov integral equation is equivalent to

$$-3\theta = C\psi. \tag{9.8-35}$$

The function θ is odd and is in L^2. Therefore it has a Fourier series of the form

$$\theta(s) = \sum_{k=1}^{\infty} a_k \sin ks. \tag{9.8-36}$$

Comparing this result with the Nekrasov integral equation and the explicit series form of the kernel (25), we see that the coefficients a_k must be

$$a_k = -\frac{1}{3\pi} \int_{-\pi}^{\pi} \frac{\sin kt}{k} \frac{\sin\theta(t)}{D + \int_0^t \sin\theta(s)ds} dt. \tag{9.8-37}$$

We next integrate (37) by parts to get

$$a_k = -\frac{1}{3\pi} \int_{-\pi}^{\pi} \cos kt \log\left[D + \int_0^t \sin\theta(s)\right] dsdt = -\frac{1}{3\pi} \int_{-\pi}^{\pi} \cos kt\, \psi(t)dt. \tag{9.8-38}$$

Therefore $-3\theta = C\psi$. This completes the proof of the lemma.

LEMMA 9.6

Let θ_D and $\sin\theta_D$ be as in (27). Then θ_D and $\sin\theta_D$ converge strongly in L^2 to θ and γ, and $\sin\theta = \gamma$.

PROOF First we show that the sequence ψ_D converges to ψ pointwise. To see this note that $\sin\theta_D$ converges weakly in L^2 to γ so that $\int_0^t \sin\theta_D(s)ds \to \int_0^t \gamma(s)ds$ for each $t\epsilon[0,\pi]$. Then, by the Lebesgue Dominated Convergence Theorem, $\psi_D \to \psi$ strongly in L^2. Since, by (35), $-3\theta_D = C\psi_D$ and since $C\psi_D \to C\psi$ strongly in L^2, we conclude that the sequence θ_D converges to θ strongly in L^2. The strong convergence of θ_D in L^2 implies that there is a subsequence of θ_D (still denoted by θ_D) that converges pointwise almost everywhere to θ. In that case $\sin\theta_D(t)$ converges to $\sin\theta(t)$ for almost all t, which in turn implies that the convergence is strong in L^2 by the Lebesgue Dominated Convergence Theorem. This proves that $\gamma = \sin\theta$ and completes the proof of the lemma.

It remains to show that the function θ actually satisfies (24) when $D = 0$. We summarize this result in the following lemma, whose proof follows in a fashion similar to those just outlined. The interested reader should refer to Nekrasov's paper for details.

LEMMA 9.7

(a) The function θ defined in (27) satisfies the Nekrasov integral equation with $D = 0$. (b) θ is discontinuous at $s = 0$ with $\sup_{s\epsilon[-\epsilon,\epsilon]} |\theta(s)| \geq \frac{\pi}{6}$. When it is known that the discontinuity at 0 is a simple jump discontinuity, then $\lim_{s\to 0^+} \theta(s) = \frac{\pi}{6}$.

This lemma provides an affirmative answer to Stokes' conjecture if it can be proved that the discontinuity at $s = 0$ is a simple jump discontinuity. This will be taken up next.

Amick, Fraenkel, and Toland compared the solutions of (24) with the solutions of the integral equation

$$\theta(x) = \frac{1}{3}\int_0^\infty k(x,y)\frac{\sin\theta(y)}{\int_0^y \sin\theta(s)ds}dy, \quad 0 < \xi < \infty, \tag{9.8-39}$$

where

$$k(x,y) = \frac{1}{\pi}\log\left(\frac{x+y}{x-y}\right). \tag{9.8-40}$$

Although the Nekrasov integral equation is not equivalent to (8-39) and (8-40), we can see the connection between them by making the following change of variables and identification in (24). Let $\xi = \tan\frac{s}{2}$ and $\psi(\xi) = \varphi(2\tan^{-1}\xi) = \varphi(s)$. Then the Nekrasov integral equation becomes

$$\psi(\xi) = \frac{1}{3}\int_0^\infty k(\xi,\eta)\frac{g(\eta)\sin\psi(\eta)}{\int_0^n g(s)\sin\psi(s)ds}d\eta, \; 0 < \xi < \infty, \tag{9.8-41}$$

where $g(\eta) = \frac{1}{1+\eta^2}$. It is the closeness of (41) and (34) that is taken advantage of in the next lemma.

LEMMA 9.8

If $\frac{\pi}{6}$ is the only solution of (1) and (2) satisfying

$$\inf_{x\in(0,\infty)} \theta(x) > 0, \qquad \sup_{x\in(0,\infty)} \theta(x) < \frac{\pi}{3}, \tag{9.8-42}$$

then any solution of the Nekrasov integral equation that satisfies

$$\liminf_{s\to 0} \varphi(s) > 0, \quad 0 < \varphi(s) \le \frac{\pi}{3} \text{ for } s \in (0, \pi)$$

also satisfies

$$\varphi(s) \to \frac{\pi}{s} \tag{9.8-43}$$

as $s \to 0$.

The proof of this lemma is quite involved and requires estimating the kernel (40) for values of ξ near zero. More importantly a detailed verification of the assumption that $\frac{\pi}{6}$ is the unique solution to (39), (40) is provided. Most of the paper is dedicated to the proof of uniqueness.

When $D = 0$ in (3.6), the Nekrasov integral operator is noncompact, and this causes the study of solutions when $D = 0$ to be quite daunting. We shall not get into a discussion of the subtle mathematical issues here but leave it to the reader to refer to the paper by Amick, Fraenkel, and Toland.

9.9 Ring Vortices[23]

In this section we will study the formation of steady vortex rings in an Euler fluid. By a vortex ring we mean a body that is homeomorphic to a solid torus. Even a casual glance at smoke rings that emanate from chimneys and winds confirm that they can be modeled as vortex rings, and by and large they are not axially symmetric. Here we shall confine our discussions to axially symmetric vortex rings. A thorough discussion of the relevant studies of vortex rings can be found in the article by Fraenkel and Berger.[24]

The first systematic study of vortex rings can be found in the famous paper of Helmholtz[25] in 1858 on the dynamics of vortex motion. He studied the motion of vortex rings of small cross-section in infinite space, with the fluid quiescent at

[23]We thank Reza Malek-Madani for his help in writing this section.

[24]L.E. Fraenkel and M.S. Berger, "A global study of steady vortex rings in an ideal fluid," *Acta Mathematica* 132 (1974): 13–51.

[25]H. Helmholtz, "Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," *J. Reine Angew. Math.* 55 (1958): 25–55.

infinity. On the basis of an approximate analysis, he showed that the ring moved, with a constant speed that was large. Kelvin[26] continued the investigation of the motion of vortex rings of small cross-sections and found an explicit expression for the speed of the ring based on an approximation. Hicks[27] and Dyson[28] also calculated the speed of vortex rings of small cross-section based on a perturbation approach.

Kelvin also formulated a variational principle concerning the motion of vortex rings. The works of Fraenkel and Berger, and Friedman and Turkington also appeal to a variational approach of the problem. We shall discuss the study of Fraenkel and Berger later.

The first study of vortex rings of arbitrarily large cross-sections was by Hill,[29] who found an exact solution. His solution corresponds to a sphere rather than a ring.

Solutions that correspond to vortex rings have been constructed by Maruhn, Norbury, Ni, Fraenkel, Friedman and Turkington, and Fraenkel and Berger. Amick and Fraenkel studied the uniqueness of Hill's vortex. Other than his study, there have been few proofs of uniqueness for vortex rings.

The problem of vortex rings shares two features with Stokes' conjecture: we have to contend with an operator that is not compact, and the nonlinearity arises due to the presence of the free surface that forms the boundary of the vortex ring. The problem lends itself to the context of the calculus of variations. We shall follow the analysis of Fraenkel and Berger in deriving the governing equations for the motion of axially symmetric vortex rings. We shall find it convenient to work in a cylindrical coordinate system.

Let us consider the axially symmetric isochoric motion of an Euler fluid. As

$$\operatorname{div} \mathbf{u} = 0$$

and we are interested in axially symmetric solutions, we can introduce the stream function q so that

$$\mathbf{u} = (\nabla q)^{\perp} \tag{6.3-2}$$

and a vector potential $\mathbf{\Phi}$ so that

$$\mathbf{\Phi} = \frac{1}{r} q(r, z) \mathbf{e}_{\theta}. \tag{9.9-1}$$

[26] W. Thompson, *Mathematical and Physical Papers*, vol. 4, (Cambridge University, 1910).

[27] W.M. Hicks, "Researches on the theory of vortex rings-part II." *Philosphical Trans. Roy Soc. London*, Series A 185 (1885): 213–245.

[28] F.W. Dyson, "The potential of an anchor ring—part II," *Philosphical Transactions of The Royal Society London*, Series A 184 (1893): 1041–1106.

[29] M.J.M. Hill, "On a spherical vortex," *Philosophical Transactions Royal Society of London*, Series A 185 (1985): 213–45.

EXERCISE 9.9.1

Let $q(r, z)$ be a sufficiently smooth function that meets (1). Show that

(i) $\operatorname{curl} \mathbf{\Phi} = \mathbf{u} = -\frac{1}{r}\frac{\partial q}{\partial z}\mathbf{e}_r + \frac{1}{r}\frac{\partial q}{\partial r}\mathbf{e}_z,$ (9.9-2)

(ii) $\operatorname{div} \mathbf{\Phi} = 0,$ (9.9-3)

(iii) $\Delta \mathbf{\Phi} = \frac{\mathbf{e}_\theta}{r} Lq,$ (9.9-4)

where

$$Lq := r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial q}{\partial r}\right) + \frac{\partial^2 q}{\partial z^2}, \tag{9.9-5}$$

and

(iv) $\mathbf{u} \cdot \nabla q = 0.$ (9.9-6)

Let P denote a half-plane

$$P := \{(r, z) | r > 0\}. \tag{9.9-7}$$

We shall consider a vortex ring of finite and constant cross-section $A \subset P$.

The equations of motion for the steady axially symmetric flow of an Euler fluid require that the vorticity remains a constant along each stream surface, and thus following Fraenkel and Berger we introduce a function $f(q)$ through

$$\omega_\theta = \lambda r f(q), \tag{9.9-8}$$

where ω_θ denotes the θ-component of the vorticity and λ is the strength of the vortex. It follows from (3) and (8) that

$$Lq = \begin{cases} \lambda r^2 f(q) & \text{in} \quad A, \\ 0 & \text{in} \quad P - \bar{A}. \end{cases} \tag{9.9-9}$$

As grad q must be continuous across the boundary of A, this constitutes one of the boundary conditions that will have to be met. Moreover, the boundary of A and the axis of symmetry must be streamlines, and thus we set

$$q\Big|_{\partial A} = 0 \tag{9.9-10}$$

and

$$q\Big|_{r=0} = -k, \tag{9.9-11}$$

where $k > 0$ determines the flux $2\pi k$ between the axis of symmetry and the boundary of the vortex ring. We shall assume that the ring moves relative to the quiescent fluid at infinity with a velocity (0,0,W). If an axis is fixed to the ring, then this implies that

$$\frac{1}{r}\frac{\partial q}{\partial z} \to 0, \ \frac{1}{\gamma}\frac{\partial q}{\partial r} \to W \text{ as } (r^2 + z^2) \to \infty. \tag{9.9-12}$$

As with the problem associated with Stokes's conjecture, we transform the governing partial differential equation into an integral equation, using the fundamental solution for the elliptic operator in (9). The fundamental solution is given by

$$F(r_0, z_0, r, z) = \frac{r_0 r}{4\pi} \int_{-\pi}^{\pi} \frac{\cos\theta d\theta}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\theta + (z - z_0)^2}}, \tag{9.9-13}$$

which satisfies

$$\Delta F = \delta(r - r_0, z - z_0) \tag{9.9-14}$$

in the sense of distributions, where δ is the Dirac measure.

Thus we seek a solution q and a boundary ∂A of A such that

$$q(r_0, z_0) = -\frac{1}{2}Wr_0^2 - k + \lambda \int_A F(r_0, z_0, r, z) f(q) r dr dz, \ q|_{\partial A} = 0. \tag{9.9-15}$$

EXERCISE 9.9.2

Show that

$$q_0(r) = \frac{1}{2}Wr^2 - k \tag{9.9-16}$$

is the stream function of a uniform flow with velocity $\mathbf{u} = -W\mathbf{e}_z$ and $L(q_0) = 0$.

Hill was able to construct an exact solution of (9) and (10), (11) and (12) when the function f was linear and k was zero. In this special case we can guess that the solution corresponds to a sphere rather than a ring. As we are dealing with a sphere it is more convenient to use spherical coordinates (R, θ, φ).

EXERCISE 9.9.3

Show that the problem given in (9)–(12), in the spherical coordinate system leads to the solution

$$q(R,\theta) = \begin{cases} \frac{1}{10}\lambda R^2 \sin^2\theta (a^2 - R^2), \ R \le a \\ -\frac{1}{2}WR^2 \sin^2\theta \left(1 - \frac{a^3}{R^3}\right), \ R \ge a, \end{cases} \tag{9.9-17}$$

where a is the radius of the sphere.

Notice that the solution for $R > a$ corresponds to the classical irrotational flow past a sphere of radius a.

The values of λ, a, and W in (11) cannot be arbitrary. Since we also need to ensure that grad q is continuous across ∂A, we find that we need to satisfy the condition $\lambda a^2 = \frac{15}{2} W$.

Hill's solution is one of the few solutions known for the problem of vortex rings. This solution corresponds to a vortex function f that is linear and for $k = 0$. It is possible to construct other solutions for $k \neq 0$ by using the implicit function theorem. However, it is far more difficult to construct solutions when f is not linear. We will address this issue next.

The main mathematical difficulty stems from the discontinuity in the right-hand side of (9) when $f(q)$ is nonlinear and the unboundedness of the domain that leads to the operator is noncompact. Fraenkel and Berger overcome this difficulty by looking at a sequence of approximations for f in bounded domains and showing by means of a very intricate analysis that a limit process leads to the solution of the original problem. This requires us to obtain estimates that are independent of the approximation process and to verify the regularity of the approximate solutions.

Fraenkel and Berger assume that the vorticity function $f(q)$ satisfies

$$f(q) \begin{cases} = 0, & q \leq 0 \\ > 0, & q > 0 \end{cases} \tag{9.9-18}$$

and furthermore that f is Hölder continuous, *i.e.*,

$$|f(q_1) - f(q_2)| \leq M|q_1 - q_2|^{\mu}, \tag{9.9-19}$$

where $\mu \in (0, 1)$. The domain is assumed to be the bounded cylinder C of radius a and length $2b$, so P is approximated by the domain

$$P_a := \{(r, z)| 0 < r < a, |z| < b\}. \tag{9.9-20}$$

Let us now define

$$\tilde{q} := q + \frac{1}{2} W r^2 + k. \tag{9.9-21}$$

Then it follows from (9) that

$$L\tilde{q} = -\lambda r^2 f(q) \text{ in } P_a. \tag{9.9-22}$$

Also, instead of prescribing the strength of the vortex $\boldsymbol{\lambda}$, we shall prescribe the kinetic energy η of the vortex, *i.e.*,

$$\int_{P_a} \frac{1}{r} \left[\left(\frac{\partial \tilde{q}}{\partial r} \right)^2 + \left(\frac{\partial \tilde{q}}{\partial z} \right)^2 \right] drdz = \eta > 0. \tag{9.9-23}$$

The condition $\eta > 0$ excludes the possibility of the trivial solution.

We shall show next that the problem (22) and (23) subject to the approximate boundary conditions reduces to maximizing a nonlinear functional over an appropriate space. Let $C_0^\infty(P_a)$ denote the set of all functions that are infinitely differentiable with compact support in P_a. We define the inner product through

$$(u, v) := \int_{P_a} \frac{1}{r^2}(u_r, v_r + u_z v_z) r\,dr\,dz. \tag{9.9-24}$$

Let $H(P_a)$ denote the completion of $C_0^\infty(P_a)$ in the norm induced by the foregoing above inner product. It then follows that $H(P_a)$ is a Hilbert space with the additional property that functions in this space have derivatives in the generalized sense and that the gradients of these functions belong to L^2. These functions also satisfy the Dirichlet boundary condition on ∂P_a in a generalized sense because each of these functions is a limit of a sequence of functions in $C_0^\infty(P_a)$. Let

$$S(\eta) : = \{u \epsilon H(P_a) \Big| \|u\|^2 = \eta\}, \tag{9.9-25}$$

$$G(q) = \int_0^q f(s)ds, \tag{9.9-26}$$

(9.9-27)

and define the functional

$$J(u) = \int_{Pa} G\left[u - \frac{1}{2}WR^2 - k\right] r\,dr\,dz. \tag{9.9-28}$$

Fraenkel and Berger prove the following theorem.

THEOREM 9.9

The variational problem $\max\limits_{u\epsilon(\eta)} J(u)$ *is equivalent to the differential formulation of governing vortex rings. Moreover, the variational problem has a solution* $\tilde{q}$ *such that* $J(\tilde{q}) > 0$ *and* $\tilde{q} \geq 0$ *almost everywhere in* P_a.

The proof is too complicated to be discussed here, and we refer readers to the paper by Fraenkel and Berger for details.

This theorem in turn guarantees a global maximizer and the existence of at least one vortex ring that satisfies the differential formulation. The question of uniqueness is an even more difficult issue. Amick and Fraenkel have carried out a meticulous study in which they prove that the solution of the spherical vortex due to Hill is unique. General uniqueness results are not available for vortex rings.

9.10 Baroclinic Flow: Theorem of V. Bjerknes

The compressible elastic fluid often fails to model well the motions of natural gases and liquids. A body of real gas at rest must be confined by a pressure that is not determined by its density alone but by its temperature and density conjointly. Such a "thermal equation of state" can be written as follows:

$$p = f(v, \alpha), \tag{9.10-1}$$

in which α is a parameter such as the temperature on some assigned scale.

Other or additional parameters may enter such an equation: the specific entropy for convenience in gas dynamics, the concentrations of the constituents of mixture, the body point X to allow for inhomogeneities or stratification. We shall not take up those possibilities.

The definition of material used heretofore and hereafter in this book, because it is purely mechanical, is not broad enough for the present purpose. In this one section we shall stretch it a little and speak of a "fluid" such that $\mathbf{T} = -p\mathbf{1}$ with p given by (1). Such a fluid is not simple and hence not elastic in the sense used in this book, but there are circumstances in which the theory of the elastic fluid applies to it. Obviously one such circumstance arises when the parameter α is assigned a constant value, as for example in a motion at constant temperature or constant specific entropy. Different such constant values produce different functions $f_\alpha(v)$ to replace $f(v, \alpha)$ and thus in general make one and the same substance behave in different circumstances like different elastic fluids. This conclusion holds more generally in circumstances where $\alpha = g(v)$. The importance of these possibilities in the atmosphere and the sea led Bjerknes to give such flows a name: *barotropic*. The theory of the elastic fluids applies to barotropic flows.

In this book we follow Bjerknes in distinguishing scrupulously the terms "flow" and "fluid." The latter is constitutive, referring to a material; the former refers to conditions in which a material body may find itself. A compressible material body may undergo an isochoric motion; all motions of an incompressible body are isochoric. All flows of an elastic fluid are barotropic, for the relation $p = f(v)$ is constitutive. The flows of the fluid defined by (1) may be barotropic but generally are not; the functions $f_\alpha(v)$ delivering barotropic flows vary with the choice of g in the restriction $\alpha = g(v)$ and hence are not constitutive.

A flow that is neither isochoric nor barotropic is called *baroclinic*; the term refers to the fact that the isochoric and isobaric surfaces decussate at each time, cutting the shape of the fluid body into a bundle of segments of unit tubes defined by the surfaces $v = 0, 1, 2, \ldots$, $p = 0, 1, 2, \ldots$, with some particular choice of units.

In his celebrated meterological studies V. Bjerknes observed that while barotropic flow is governed by the classical theorems on the permanence of vorticity (8.6), baroclinic flow may create and destroy circulation.

EXERCISE 9.10.1
Show that the rate of change of the circulation $C(\mathcal{C})$ of a simple material circuit $\mathcal{C}$ bounding a simply connected surface a, say $\mathbf{x} = \mathbf{f}(a, b)$, is given by

$$\dot{C}(\mathcal{C}) = \int_{\mathrm{a}} \operatorname{skw} \operatorname{grad} \ddot{\mathbf{x}} \cdot \partial_a \mathbf{f} \wedge \partial_b \mathbf{f} \, da \, db, \tag{9.10-2}$$

while Euler's dynamical equation (8.1-1) when $\mathbf{b}$ is lamellar yields

$$\operatorname{skw} \operatorname{grad} \ddot{\mathbf{x}} = (1/2) \operatorname{grad} v \wedge \operatorname{grad} p. \tag{9.10-3}$$

We now suppose that for the independent parameters a and b in terms of which a is expressed we may choose p and v; while never possible in barotropic flow, such is the case in a sufficiently small region of baroclinic flow. Then putting (3) into (2) reduces the integrand to 1 and hence delivers the *circulation theorem* of V. Bjerknes for flows described in an inertial framing:

$$\dot{C}(\mathcal{C}) = \int_{a_{pv}} dp dv = A(a_{pv}), \tag{9.10-4}$$

in which a_{pv} denotes the projection of a onto its support in the $p - v$ plane, while $A(a_{pv})$ is the signed area of a_{pv}. Bjerknes expressed this statement much as follows: *The rate of increase of the circulation of* $\mathcal{C}$ *induced by baroclinic flow is the number of unit* p-v *tubes that* $\mathcal{C}$ *embraces. The portions of* a *in which the smaller angle between* grad p *and* grad v *is less than a right angle increase the circulation counterclockwise in the* p-v *plane; if the samller angle is greater than a right angle, the circulation increases clockwise.*

The most interesting applications of this theorem were noticed by Bjerknes himself, and his papers still make excellent reading.[30] Consider first the seashore, and suppose that, as usual, grad v points upward or nearly so. A quiet sea may be regarded as a body that preserves its temperature unchanged, day and night; the sun warms the surface of the shore by day, while by night the sun's absence lets the surface of the shore cool off through its contact with a layer of earth beneath, the temperature of which remains little affected by the cycle of night and day. By day then, grad θ points from sea toward land, while by night its sense is opposite.

[30]The two applications described and several more may be found in V. Bjerknes, "Das dynamische Prinzip der Zirkulationsbewegungen in der Atmosphäre," *Meterologische Zeitschrift* 17 (1900): 97–106, 145–156.

Thus grad p has a component $(\partial_\theta f)$ grad θ that points toward land by day, toward sea by night. According to Bjerknes's theorem, should the atmosphere be still in the daytime a cool breeze would begin to flow landward near the surface, turning back from land to sea at higher altitudes. At night just the reverse would happen, and a warm breeze would blow out to sea. The rate of increase in circulation at each instant may be calculated from (3).

The same observations may be applied to the global winds. The poles act as reservoirs of cold, while the sun's continual warmth heats the equatorial regions. Bjerknes's theorem provides continual increase in the circulation from the poles toward the equator at low altitudes. Thus the prevailing winds should be north winds in the northern hemisphere, south winds in the southern hemisphere. The resulting global circulation would be confined, subject to disturbances, to meridional planes. Because such is not the case in reality, Bjerknes considered the effects of the earth's spin, assumed steady, upon the circulation of a material curve in the atmosphere.[31] In the general theorem we regard a circuit $\mathcal{C}$ in terms both of an inertial framing and of a terrestrial framing, the two instantaneously coinciding; that is, $\mathbf{Q} = \mathbf{1}$ in (2.3-8). Using ω to denote the angular speed of the earthbound framing with respect to the inertial framing, we obtain (in an obvious notation)

$$C_{\text{iner}}(\mathcal{C}) - C_{\text{terr}}(\mathcal{C}) = 2\omega A_{\text{eq}}(\mathcal{C}), \tag{9.10-5}$$

in which $A_{\text{eq}}(\mathcal{C})$ stands for the area bounded by the projection of $\mathcal{C}$ onto the earth's equatorial plane. Now assuming that $\omega = \text{const.}$, by differentiating (5) and combining it with (4) we conclude that

$$\dot{C}_{\text{terr}}(\mathcal{C}) = A(a_{pv}) - 2\omega \dot{A}_{\text{eq}}(\mathcal{C}). \tag{9.10-6}$$

This statement is Bjerknes's *general circulation theorem* expressed for use in a framing in which the earth is at rest. The first quantity on the right-hand side shows how baroclinic flow creates circulation; the second, how decrease or increase of $A_{\text{eq}}(\mathcal{C})$ contributes to augment or diminish the circulation, regardless of how that circulation is produced.

We now return to the prevailing winds induced by the difference of temperatures at the poles and the equator and recall that the effect of baroclinic flow by itself is to produce circulation around material curves lying in meridional planes. The circulation in a quadrilateral bounded by meridians and parallels could then be null and would remain null if the speeds in the two meridianal parts were equal at each parallel. The second term in (6) shows that a non-null circulation would then arise. Thus a mechanism for creating non-null components of winds in east–west directions is provided.

Interpretations of Bjerknes's theorem are necessarily cautious. It does not provide solutions of Euler's dynamical equation. Referring only to tendencies, the interpretations exhibit circumstances in which a change of circulation must

[31] V. Bjerknes, "Zirkulation relativ zu der Erde," *Meteorologische Zeitschrift* 19 (1902): 97–108.

result. Much as Helmholtz's theorems on the preservation of vorticity make the convective aspects of rotational flows "approachable in concept," Bjerknes's theorem makes approachable in concept also the creation and destruction of vorticity in dissipationless fluid bodies.

10

Compressible Euler Fluids

10.1 Barotropic Fluids

Thus far, we have mainly confined our attention to incompressible fluids. While the assumption of incompressibility is reasonable when we restrict our attention to liquids, it is inappropriate when dealing with gases. In this section, we shall discuss briefly some interesting features that are a consequence of compressibility. The main results presented in this section pertain to the Munk-Prim substitution principle and the uniqueness of the flow of a compressible fluid. A detailed treatment of various aspects of the mechanics of compressible fluids can be found in Prandtl and Tietjens,[1] Von Mises and Friedrichs,[2] Courant and Friedrichs[3] and Landau and Lifshitz.[4]

Recall that the constitutive equation for an Euler fluid is

$$\mathbf{T} = -p(\rho)\mathbf{1}. \tag{4.1-8}$$

We shall find it convenient to write Euler's dynamical equation (9.1-1) in the modified form

$$\rho\ddot{\mathbf{x}} = \frac{-\partial p}{\partial \rho}(\text{grad}\,\rho) + \rho\mathbf{b}. \tag{10.1-1}$$

In this book we have considered only the mechanical response of fluids and diligently avoided discussions from a thermodynamical standpoint. This becomes particularly onerous when we deal with a compressible fluid, as the pressure depends on both the density and the temperature of the fluid. However, in keeping

[1] L. Prandtl and O.G. Tietjens, *Fundamentals of Hydro- and Aero-mechanics and Applied Hydro- and Aero-mechanics* (New York: McGraw-Hill, 1934).

[2] R. Von Mises and K.O. Friedrichs, *Fluid Dynamics*, (New York: Springer-Verlag 1971).

[3] R. Courant and K.O. Friedrichs, *Supersonic Flow and Shock Waves* (New York: Interscience, 1948).

[4] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Oxford and New York: Pergamon Press, 1987).

with the intent and spirit of the book, we shall refrain from discussing the response of compressible fluids from a thermodynamical point of view.

We shall call a compressible fluid *barotropic* if the pressure is related to the density through

$$p = k\rho^{\gamma}, \tag{10.1-2}$$

where k is a constant and $\gamma \geq 1$, γ being a constant. Air at room temperature can be approximately described by a relation of the form (2) with $\gamma = 1.4$. An ideal gas, within the context of a purely mechanical point of view corresponds to (2) with $\gamma = 1$.

For barotropic gases, (1) reduces to

$$-k\gamma\rho^{\gamma-2}(\text{grad}\,\rho) = \ddot{\mathbf{x}} - \mathbf{b}. \tag{10.1-3}$$

Also, from (7.1-2) and (2), we find that

$$R = \left(\frac{k\gamma}{\gamma - 1}\right)\rho^{\gamma-1}. \tag{10.1-4}$$

We shall see later that the quantity

$$c := \left(\frac{dp}{d\rho}\right)^{1/2}, \tag{10.1-5}$$

denotes the speed of sound in a compressible fluid, and it follows from (4) and (5) that

$$R = \frac{c^2}{\gamma - 1}. \tag{10.1-6}$$

In defining (5) we have of course presumed that

$$\frac{dp}{d\rho} > 0, \tag{10.1-7}$$

the implications of which will be discussed in more detail later. By (6), (9.6-4) reduces to

$$\frac{c^2}{\gamma - 1} + \frac{1}{2}|\dot{\mathbf{x}}|^2 + \varpi + Q = k. \tag{10.1-8}$$

10.2 Irrotational Motion

Let us consider the irrotational motion of a compressible Euler fluid subject to a conservative body-force field. In this case, we can introduce the velocity potential P_v through

$$\dot{\mathbf{x}} = -\,\text{grad}\,P_v, \tag{2.2-36}$$

and Euler's Dynamical equation yields

$$\frac{\partial P_{\mathrm{v}}}{\partial t} - \frac{1}{2} |\operatorname{grad} P_{\mathrm{v}}|^2 - R - \varpi = 0. \tag{9.4-2}$$

Taking the material time derivative of (8.4-1) and using the chain rule, we obtain

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial P_{\mathrm{v}}}{\partial t}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|\operatorname{grad} P_{\mathrm{v}}|^2\right) - \frac{\mathrm{d}\varpi}{\mathrm{d}t} = 0. \tag{10.2-1}$$

Next, we observe that by virtue of (10.1-5)

$$\begin{aligned} \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{c^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \\ &= \frac{c^2}{\rho}\left[\frac{\partial \rho}{\partial t} + \operatorname{grad}\rho \cdot \operatorname{grad} P_{\mathrm{v}}\right] \\ &= -c^2 \Delta P_{\mathrm{v}}, \end{aligned} \tag{10.2-2}$$

Where the last equality is a consequence of the conservation of mass. It follows from (1) and (2) that

$$\begin{aligned} &c^2 \Delta P_{\mathrm{v}} + \frac{\partial \varpi}{\partial t} - (\operatorname{grad}\varpi \cdot \operatorname{grad} P_{\mathrm{v}}) \\ &+ \frac{1}{2}\frac{\partial}{\partial t}[|\operatorname{grad} P_{\mathrm{v}}|^2] - \operatorname{grad}\left[\frac{1}{2}|\operatorname{grad} P_{\mathrm{v}}|^2\right] \cdot [\operatorname{grad} P_{\mathrm{v}}] \\ &- \frac{\partial^2 P_{\mathrm{v}}}{\partial t^2} + [\operatorname{grad} P_{\mathrm{v}}] \cdot \left[\operatorname{grad}\left(\frac{\partial}{\partial t} P_{\mathrm{v}}\right)\right] = 0. \end{aligned} \tag{10.2-3}$$

In this equation, ϖ is given and c^2 is a known function of ρ. Since R is a known function of ρ by (10.1-7), c^2 can be expressed as a function of $\partial P_{\mathrm{v}}/\partial t$, $|\operatorname{grad} P_{\mathrm{v}}|^2$ and ϖ. Thus, (3) is a partial differential equation in P_{v}. In the case of an incompressible fluid, $c = \infty$ and (3) reduces to

$$\Delta P_{\mathrm{v}} = 0. \tag{2.2-37}$$

In the case of steady flow, in the absence of the body-force, we have

$$c^2 \Delta P_{\mathrm{v}} - \operatorname{grad}[|\operatorname{grad} P_{\mathrm{v}}|^2] \cdot [\operatorname{grad} P_{\mathrm{v}}] = 0. \tag{10.2-4}$$

A special case of this equation was first given by Lagrange.

Next, we observe from (10.1-7) that, in the case of steady flows in the absence of body-forces, the pressure must decrease for velocity to increase. This leads to an upper bound for the velocity of a particle along a streamline, namely that which corresponds to the value of the velocity when the particle is discharged into vacuum. We also observe that the density is also a decreasing function of the velocity since

$$\frac{d(\log \rho)}{d(\log |\operatorname{grad} P_v|)} = \frac{-|\dot{\mathbf{x}}|^2}{c^2}. \tag{10.2-5}$$

We define the *Mach number* M at any point in the fluid through

$$M := \frac{|\dot{\mathbf{x}}|}{c}, \tag{10.2-6}$$

and thus when $M < 1$ the flow is subsonic, while if $M > 1$ the flow is supersonic.

10.3 Munk-Prim Substitution

We now inquire into the possibility of more than one flow having the same streamline, Mach number, and pressure distribution but different density distribution and magnitude of velocity. This problem was solved by Prim.

Let us introduce a vector field $\mathbf{s}$, parallel to the velocity field, of the form

$$\mathbf{s} = \sqrt{\rho}\dot{\mathbf{x}}. \tag{10.3-1}$$

We shall consider steady flows. Then, the conservation of mass (1.5-6) can be written in terms of $\mathbf{s}$ as

$$\operatorname{div}\mathbf{s} + \frac{1}{2}(\mathbf{s}\cdot\operatorname{grad}(\log\rho)) = 0, \tag{10.3-2}$$

while the modified Euler's dynamical equation (10.1-1) becomes

$$\operatorname{grad} p + \frac{1}{2}[\mathbf{s}\cdot\operatorname{grad}(\log\rho)]\mathbf{s} + [\operatorname{grad}\mathbf{s}]\mathbf{s} = 0. \tag{10.3-3}$$

Eliminating the density from (2) and (3), we have

$$[\operatorname{grad}\mathbf{s}]\mathbf{s} + (\operatorname{div}\mathbf{s}) + \operatorname{grad} p = 0. \tag{10.3-4}$$

Forming the scalar product of (4) with $\mathbf{s}$, we obtain

$$(\operatorname{div}\mathbf{s})|\mathbf{s}|^2 + \frac{1}{2}\mathbf{s}\cdot\operatorname{grad}(|\mathbf{s}|^2) + \mathbf{s}\cdot\operatorname{grad} p = 0. \tag{10.3-5}$$

An elementary application of the chain rule and the use of (10.2-6) yields

$$\mathbf{s}\cdot\operatorname{grad}(\log\rho) = \frac{-M^2}{|\mathbf{s}|^2}\mathbf{s}\cdot\operatorname{grad}\rho. \tag{10.3-6}$$

It then follows from (5) and (6) that

$$\mathbf{s}\cdot\operatorname{grad}(\log\rho) = -M^2(\operatorname{div}\mathbf{s}) - \frac{M^2}{|\mathbf{s}|^2}\mathbf{s}\cdot\operatorname{grad}(|\mathbf{s}|^2). \tag{10.3-7}$$

Combining (5) and (7), we obtain

$$\operatorname{div}\mathbf{s} + \frac{M^2}{(M^2-2)|\mathbf{s}|^2}\mathbf{s}\cdot\operatorname{grad}(|\mathbf{s}|^2) = 0. \tag{10.3-8}$$

Now, (4) and (8) are completely equivalent to the conservation of mass and the balance of linear momentum. Here we have managed to eliminate the density ρ from the equations, and as long as p, $\mathbf{s}$, and M are the same, we have a solution regardless of the value of ρ. However, if the full thermodynamic problem is considered, we have to satisfy the energy equation in which the density ρ appears, and in this case it can be shown that if p, $\mathbf{s}$, and M is a solution, then p, $\mathbf{s}$, $m\rho$ and M is also a solution, where m is a constant along the streamline. If in two flows the densities are ρ and $m\rho$, then the velocities in the two flows would be $\dot{\mathbf{x}}$ and $\frac{\dot{\mathbf{x}}}{\sqrt{m}}$, respectively. Since m may be any function that is a constant along a streamline, it can be chosen in such a fashion that any other flow variable is constant throughout the flow. Our analysis shows that there is a multiplicity of solutions possible for the equations governing the flow of a compressible Euler fluid. In the next section we show that given specific boundary and initial conditions and certain assumptions regarding the uniformity of the flow variables and their gradients in time and the boundedness of the flow variables, we can prove uniqueness of the solutions.

10.4 Uniqueness

We now proceed to discuss the simple but ingenious analysis of Graffi[5] regarding the uniqueness of flows of a compressible fluid. In recent years, there have been numerous sophisticated studies devoted to proving existence and uniqueness of solutions, under far weaker assumptions. Uniqueness, for small data for the steady flow of a compressible Euler fluid, has been proved under very weak assumptions by Beirão Da Veiga.[6]

Following Graffi we shall consider a slightly more general form of the Euler's dynamical equation (10.1-1), namely

$$\rho\ddot{\mathbf{x}} = \frac{-\mathrm{d}p}{d\rho}(\operatorname{grad}\rho) + \mathbf{F}(\rho). \tag{10.4-1}$$

In fact, Graffi considers the case of a compressible Navier-Stokes fluid, while here we shall consider only the Euler fluid. We recall that the boundary condition that is appropriate to an Euler fluid at a stationary wall is

$$\mathbf{n}\cdot\dot{\mathbf{x}} = 0. \tag{1.8-5}$$

[5]D. Graffi, "Il teorema di unicitá nella dinamica dei fluidi compressibili," *Journal of Rational Mechanics Analysis* 2 (1953): 99–106.

[6]H. Beirão da Veiga, "An L^p theory for n-dimensional stationary, compressible Navier-Stokes equations and the incompressible limit for compressible fluids: The equilibrium solution," *Computational Mathematics and Mathematical Physics* 109 (1987): 229–248.

We shall assume that the flow domain is bounded by a rigid wall so that we do not have a free boundary. The results presented here hold even under the weaker condition that

$$\mathbf{n} \cdot \dot{\mathbf{x}} \geq 0,$$

where $\mathbf{n}$ is the unit outward normal to the boundary. We shall not consider the weaker case here, though the results can be extended with no difficulty.

Let $(\bar{\mathbf{v}}, \bar{\rho})$ and $(\hat{\mathbf{v}}, \hat{\rho})$ denote two velocity-density pairs that satisfy the conservation of mass and the balance of linear momentum. Let

$$\mathbf{u} := \hat{\mathbf{v}} - \bar{\mathbf{v}}, \tag{10.4-2}$$

$$\rho := \hat{\rho} - \bar{\rho}. \tag{10.4-3}$$

It is straightforward to show that

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \bar{\mathbf{v}}) + \operatorname{div}[(\bar{\rho} + \rho)\mathbf{u}] = 0, \tag{10.4-4}$$

$$\begin{aligned} &\rho \frac{\partial(\mathbf{u} + \bar{\mathbf{v}})}{\partial t} + \bar{\rho}\frac{\partial \mathbf{u}}{\partial t} + \rho[\operatorname{grad}(\bar{\mathbf{v}} + \mathbf{u}) \\ &\quad + \bar{\rho}[\operatorname{grad} \mathbf{u}]](\bar{\mathbf{v}} + \mathbf{u}) + \bar{\rho}[\operatorname{grad} \bar{\mathbf{v}}]\mathbf{u} \\ &= -p'(\bar{\rho} + \rho)[\operatorname{grad} \rho] - [p'(\bar{\rho} + \rho) - p'(\bar{\rho})] \operatorname{grad} \bar{\rho} \\ &\quad + \mathbf{F}(\bar{\rho} + \bar{\rho}) - \mathbf{F}(\bar{\rho}), \end{aligned} \tag{10.4-5}$$

where $p' = \frac{dp}{d\rho}$. The boundary condition (1.8-5) becomes

$$\mathbf{n} \cdot \bar{\mathbf{v}} = \mathbf{n} \cdot \hat{\mathbf{v}} = \mathbf{n} \cdot \mathbf{u} = 0 \text{ on } \partial V \times [0, \infty) \tag{10.4-6}$$

On forming the scalar product of (4) with $\mathbf{u}$ and integrating over the flow domain V, we obtain

$$\begin{aligned} &\int_V \rho \frac{\partial(\bar{\mathbf{v}} + \mathbf{u})}{\partial t} \cdot \mathbf{u}\, dv + \int_V \frac{\bar{\rho}}{2} \frac{\partial |\mathbf{u}|^2}{\partial t} dv + \int_V \rho[\operatorname{grad}(\bar{\mathbf{v}} + \mathbf{u})](\bar{\mathbf{v}} + \mathbf{u}) \cdot \mathbf{u} dv \\ &+ \int_V \bar{\rho}[\operatorname{grad}(\mathbf{u})](\bar{\mathbf{v}} + \mathbf{u}) \cdot \mathbf{u} dv + \int_V \bar{\rho}[\operatorname{grad}(\bar{\mathbf{v}})]\mathbf{u} \cdot \mathbf{u} dv \\ &+ \int_V p'(\bar{\rho} + \rho) \operatorname{grad} \rho \cdot \mathbf{u} dv + \int_V [p'(\bar{\rho} + \rho) - p'(\bar{\rho})] \operatorname{grad} \bar{\rho} \cdot \mathbf{u} dv \\ &- \int_V [\mathbf{F}(\bar{\rho} + \rho) - \mathbf{F}(\bar{\rho})] \cdot \mathbf{u} dv = 0. \end{aligned} \tag{10.4-7}$$

Suppose $|\frac{\partial}{\partial t}(\bar{\mathbf{v}} + \mathbf{u})|$ is bounded by N_1 for all time and at all points in the flow domain. Then

$$\int_V \rho \frac{\partial(\bar{\mathbf{v}} + \mathbf{u})}{\partial t} \cdot \mathbf{u} dv \leq N_1 \int_V \rho |\mathbf{u}| dv \leq \frac{N_1}{2} \int_V [\rho^2 + |\mathbf{u}|^2] dv, \tag{10.4-8}$$

the last inequality holding by virtue of $ab \leq \frac{1}{2}(a^2+b^2)$. Similarly, if N_2 represents the least upper bound for $|[\text{grad}(\bar{\mathbf{v}}+\mathbf{u})](\bar{\mathbf{v}}+\mathbf{u})|$ in $V \times [0,\infty)$, then

$$\int_V \rho[\text{grad}(\bar{\mathbf{v}}+\mathbf{u})](\bar{\mathbf{v}}+\mathbf{u})\cdot\mathbf{u}\,dv \leq \frac{N_2}{2}\int_V [\rho^2+|\mathbf{u}|^2]dv. \tag{10.4-9}$$

Next, it is quite straightforward to show that

$$\int_V \bar{\rho}[\text{grad}\,\mathbf{u}](\bar{\mathbf{v}}+\mathbf{u})\cdot\mathbf{u}\,dv = -\frac{1}{2}\int_V \text{div}[|\mathbf{u}|^2\bar{\rho}(\bar{\mathbf{v}}+\mathbf{u})]dv + \frac{1}{2}\int_V |\mathbf{u}|^2\,\text{div}[\bar{\rho}(\bar{\mathbf{v}}+\mathbf{u})]dv. \tag{10.4-10}$$

Applying the divergence theorem and appealing to (6), we obtain

$$\int_V \bar{\rho}[\text{grad}\,\mathbf{u}](\bar{\mathbf{v}}+\mathbf{u})\cdot\mathbf{u}\,dv = \frac{1}{2}\int_V |\mathbf{u}|^2\,\text{div}[\bar{\rho}(\bar{\mathbf{v}}+\mathbf{u})]dv. \tag{10.4-11}$$

If N_3 denotes the least upper bound of $\text{div}[\bar{\rho}(\bar{\mathbf{v}}+\mathbf{u})]$ in $V \times [0,\infty)$, we have

$$\int_V \bar{\rho}[\text{grad}\;\mathbf{u}](\bar{\mathbf{v}}+\mathbf{u})\cdot\mathbf{u}\,dv \leq \frac{N_3}{2}\int_V |\mathbf{u}|^2 dv. \tag{10.4-12}$$

Since $\bar{\mathbf{v}}$ has bounded derivatives and we have assumed that the density is bounded, there exists an N_4 such that

$$\int_V \bar{\rho}[\text{grad}\;\bar{\mathbf{v}}]\mathbf{u}\cdot\mathbf{u}\,dv \leq N_4\int_V |\mathbf{u}|^2 dv. \tag{10.4-13}$$

Next, a simple computation yields

$$p'(\bar{\rho}+\rho)\,\text{grad}\,\rho\cdot\mathbf{u} = \text{div}[(p'(\bar{\rho}+\rho))\rho\mathbf{u}] - \rho\;\text{grad}[p'(\bar{\rho}+\rho)]\cdot\mathbf{u} - p'(\bar{\rho}+\rho)\rho\,\text{div}\,\mathbf{u}. \tag{10.4-14}$$

Substituting for div**u** from (4) and simplifying, we obtain

$$\begin{aligned} p'(\bar{\rho}+\rho)\,\text{grad}\,\rho\cdot\mathbf{u} = {} & \text{div}[(p'(\bar{\rho}+\rho))\rho\mathbf{u}] + \frac{1}{2}\frac{p'(\bar{\rho}+\rho)}{(\bar{\rho}+\rho)}\frac{\partial(\rho)^2}{\partial t} \\ & - \rho\left[\text{grad}\;p'(\bar{\rho}+\rho) - \frac{p'(\bar{\rho}+\rho)}{(\bar{\rho}+\rho)}\text{grad}(\bar{\rho}+\rho)\right]\cdot\mathbf{u} \\ & + \frac{p'(\bar{\rho}+\rho)}{(\bar{\rho}+\rho)}\rho^2\,\text{div}\,\bar{\mathbf{v}} + \frac{p'(\bar{\rho}+\rho)}{(\bar{\rho}+\rho)}\text{grad}\,\rho\cdot\bar{\mathbf{v}}. \end{aligned} \tag{10.4-15}$$

A lengthy but straightforward calculation gives

$$\begin{aligned} p'(\bar{\rho}+\rho)\,\text{grad}\,\rho\cdot u = {} & \frac{1}{2}\frac{\partial}{\partial t}\left[\frac{p'(\bar{\rho}+\rho)}{(\bar{\rho}+\rho)}\rho^2\right] - \frac{\rho^2}{2}\frac{\partial}{\partial t}\left[\frac{p'(\bar{\rho}+\rho)}{(\bar{\rho}+\rho)}\right] \\ & + \text{div}[(p'(\bar{\rho}+\rho))\rho\mathbf{u}] - \rho[\text{grad}\,p'(\bar{\rho}+\rho) - \frac{p'(\bar{\rho}+\rho)}{(\bar{\rho}+\rho)}\text{grad}(\bar{\rho}+\rho)]\cdot\mathbf{u} \end{aligned}$$

$$+\frac{\rho^2}{2}\frac{p'(\bar\rho+\rho)}{(\bar\rho+\rho)}\operatorname{div}\bar{\mathbf{v}}+\frac{1}{2}\operatorname{div}\left[\frac{p'(\bar\rho+\rho)}{(\bar\rho+\rho)}\rho^2\bar{\mathbf{v}}\right]-\frac{\rho^2}{2}\operatorname{grad}\left[\frac{p'(\bar\rho+\rho)}{(\bar\rho+\rho)}\right]\cdot\bar{\mathbf{v}}. \tag{10.4-16}$$

If N_5 and N_6 denote the least upper bounds for

$$|\operatorname{grad} p'(\bar\rho+\rho)-\frac{p'(\bar\rho+\rho)}{(\bar\rho+\rho)}\operatorname{grad}(\bar\rho+\rho)|$$

and

$$\left|-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{p'(\bar\rho+\rho)}{(\bar\rho+\rho)}\right)+\frac{1}{2}\frac{p'(\bar\rho+\rho)}{(\bar\rho+\rho)}\operatorname{div}\bar{\mathbf{v}}-\frac{1}{2}\operatorname{grad}\left[\frac{p'(\bar\rho+\rho)}{(\bar\rho+\rho)}\right]\cdot\bar{\mathbf{v}}\right|$$

respectively in $V\times[0,\infty)$, then on integrating (15) over V, using the divergence theorem and appealing to the boundary condition (6), we obtain

$$-\int_V p'(\bar\rho-\rho)\operatorname{grad}\rho\cdot\mathbf{u}dv\le-\frac{1}{2}\int_V\frac{\partial}{\partial t}\left[\frac{p'(\bar\rho-\rho)}{(\bar\rho-\rho)}\rho^2\right]dv$$
$$+\frac{N_5}{2}\int_V|\mathbf{u}|^2dv+\left(\frac{N_5}{2}+N_6\right)\int_V\rho^2dv. \tag{10.4-17}$$

If N_7 denotes the least upper bound for $|\frac{\partial\mathbf{F}}{\partial\rho}|$ for the density varations between its extreme values, then appealing to the mean value theorem we have

$$|[\mathbf{F}(\bar\rho+\rho)-\mathbf{F}(\bar\rho)]\cdot\mathbf{u}|\le N_7(\rho^2+|\mathbf{u}|^2), \tag{10.4-18}$$

and thus

$$\int_V[\mathbf{F}(\bar\rho+\rho)-\mathbf{F}(\rho)]\cdot\mathbf{u}dv\le N_7\int_V(\rho^2+|\mathbf{u}|^2)dv. \tag{10.4-19}$$

Next, suppose $|\frac{\partial\bar\rho}{\partial t}|$ is bounded above by a constant N_8. Then

$$\int_V\frac{\partial\bar\rho}{\partial t}|\mathbf{u}|^2dv\le N_8\int_V|\mathbf{u}|^2dv. \tag{10.4-20}$$

Finally, if $|p''(\rho)|$ and $|\operatorname{grad}\rho|$ are bounded above, then there exists a constant N_9 such that

$$\int_V[p'(\bar\rho+\rho)-p'(\bar\rho)]\operatorname{grad}\bar\rho\cdot\mathbf{u}dv\le N_9\int_V(\rho^2+|\mathbf{u}|^2)dv. \tag{10.4-21}$$

It then follows from (7)-(21) that there are constants M_1 and M_2 such that

$$\frac{1}{2}\int_V\frac{\partial}{\partial t}(\bar\rho|\mathbf{u}|^2)dv+\frac{1}{2}\int_V\frac{\partial}{\partial t}\left[\frac{p'(\bar\rho+\rho)}{(\bar\rho+\rho)}\rho^2\right]dv\le M_1\int_V|\mathbf{u}|^2dv+M_2\int_V\rho^2dv. \tag{10.4-22}$$

Suppose that initially at $t=0$, the velocities $\bar{\mathbf{v}}$ and $\hat{\mathbf{v}}$ and the densities $\bar\rho$ and $\hat\rho$ are the same. Then

$$\mathbf{u}(\mathbf{x},0)=\mathbf{0},\qquad\forall\mathbf{x}\in V, \tag{10.4-23}$$

and

$$\rho(\mathbf{x},0)=0, \qquad \forall\mathbf{x}\in V. \tag{10.4-24}$$

Integrating (22) between 0 and t, and using (23) and (24), we obtain

$$\frac{1}{2}\int_V \rho|\mathbf{u}|^2 dv+\frac{1}{2}\int_V \frac{p'(\bar{\rho}+\rho)}{(\bar{\rho}+\rho)}\rho^2 dv \leq \int_0^t M_1 \int_V |\mathbf{u}|^2 dv dt+\int_0^t M_2 \int_V \rho^2 dv dt. \tag{10.4-25}$$

If we suppose that $\bar{\rho}$ is bounded from below by a positive constant m, *i.e.*,

$$\bar{\rho}\geq m>0, \tag{10.4-26}$$

and $p'>0$, then there is a greatest lower bound n for $\frac{p'(\bar{\rho}+\rho)}{(\bar{\rho}+\rho)}$ in $V\times[0,\infty)$ such that

$$\frac{m}{2}\int_V \rho|\mathbf{u}|^2 dv+\frac{n}{2}\int_V \rho^2 dv \leq M\int_0^t\int_V(|\mathbf{u}|^2+\rho^2)dv dt, \tag{10.4-27}$$

where M is the larger of the two numbers M_1 and M_2. It immediately follows from (27) that there is a positive constant $\bar{M}$ such that

$$\int_V(|\mathbf{u}|^2+\rho^2)dv \leq \bar{M}\int_0^t\int_V(|\mathbf{u}|^2+\rho^2)dv dt. \tag{10.4-28}$$

Since (28) is true for arbitrary values of t, it immediately follows that

$$\int_v(|\mathbf{u}|^2+\rho^2)dv=0, \tag{10.4-29}$$

and thus we conclude that $\mathbf{u}=\mathbf{0}$ and $\rho=0$, implying uniqueness of solutions.

The results we have obtained presumes that the gradients of the velocity field and the density and its time derivative are bounded for all times, as well as several other boundedness assumptions. Uniqueness of solutions has been obtained under far weaker assumptions.

10.5 Linear Waves

We now proceed to linearize (10.1-1), which leads to a wave equation and allows us to discuss the significance of the wave speed c defined in (10.1-5). Let p_e and ρ_e denote the static equilibrium values of the pressure and density, respectively. Let us consider a perturbance to the static equilibrium of the compressible fluid, of the form

$$p=p_e+\varepsilon\hat{p}, \quad \rho=\rho_e+\varepsilon\hat{\rho}, \quad \dot{\mathbf{x}}=\varepsilon\hat{\mathbf{v}}, \tag{10.5-1}$$

where $\varepsilon \ll 1$. Now equations (8.1-1) and (1.4-5) at $O(\varepsilon)$ lead to

$$\frac{\partial \rho}{\partial t} + \rho_e \operatorname{div} \hat{\mathbf{v}} = 0, \tag{10.5-2}$$

and

$$\rho_e \frac{\partial \hat{\mathbf{v}}}{\partial t} = -\operatorname{grad}(\hat{p}). \tag{10.5-3}$$

Notice that (2) and (3) are linear.

If p depends smoothly on ρ, then, expanding p about ρ_e in a Taylor's series, we have

$$\begin{aligned} p(\rho) &= p(\rho_e) + \left[\frac{dp(\rho_e)}{d\rho}\right](\rho - \rho_e) + O(\rho - \rho_e)^2 \\ &= p_e + \left[\frac{dp(\rho_e)}{d\rho}\right]\hat{\rho} + O(\varepsilon^2), \end{aligned} \tag{10.5-4}$$

and thus by (1), neglecting terms of order ε^2, we get

$$\hat{p} = \left[\frac{dp(\rho_e)}{d\rho}\right]\hat{\rho}. \tag{10.5-5}$$

If the flow is irrotational and the region simply connected, then

$$\dot{\mathbf{x}} = -\operatorname{grad} P_{\mathrm{v}}. \tag{2.2-36}$$

Next, it follows from (10.2-3) and (2.2-36) that

$$\hat{p} = -\rho_e \frac{\partial}{\partial t}(P_{\mathrm{v}}). \tag{10.5-6}$$

Using (5) and (6), we get

$$\hat{\rho} = \frac{1}{\left[\frac{dp(\rho_e)}{d\rho}\right]}\left\{\rho_e \frac{\partial P_{\mathrm{v}}}{\partial t}\right\}. \tag{10.5-7}$$

Next, substituting (2.2-36) and (7) into (2), we obtain

$$\frac{\partial^2 P_{\mathrm{v}}}{\partial t^2} - \left[\frac{dp(\rho_e)}{d\rho}\right]\Delta P_{\mathrm{v}} = 0. \tag{10.5-8}$$

Thus the perturbance of the form (1) leads to a wave equation, whose speed is given by $\left[\frac{dp(\rho_e)}{d\rho}\right]^{\frac{1}{2}}$, thereby justifying our definition (5) for c. We observe that by virtue of (6) and (8), $\hat{p}$ also obeys a wave equation, and consequently by (10.1-2), ρ also obeys a wave equation.

10.6 Steady Flow in an Axially Symmetric Tube

We shall consider the steady flow of a compressible Euler fluid in an axially symmetric straight tube with its axis along the x_1 direction. Let the cross-sectional area of the tube be denoted by $A(x_1)$. We shall suppose that

$$\dot{\mathbf{x}} = v(x_1)\mathbf{i}, \quad \rho = \rho(x_1), \tag{10.6-1}$$

and once again we shall denote x_1 by x.

Conservation of mass (1.4-5) immediately yields

$$\rho v A = K_1 = \text{const.}, \tag{10.6-2}$$

where K_1 is the mass flux. Equation (2) implies that

$$\frac{dv}{v} = \frac{d\rho}{\rho} = \frac{dA}{A}. \tag{10.6-3}$$

Euler's dynamical equation (10.1-1) leads to

$$\rho\frac{dv}{dt} = -\operatorname{grad}(p + \varpi). \tag{10.6-4}$$

For a barotropic fluid, a simple calculation using (10.1-5) and (4) shows that

$$v dv = -c^2\frac{d\rho}{\rho}. \tag{10.6-5}$$

On comparing (3) and (5), we obtain

$$\left(\frac{v^2}{c^2} - 1\right)\frac{dv}{v} = \frac{dA}{A}. \tag{10.6-6}$$

It follows from (6) that if $A(x)$ does not have a local extremum, then v cannot have a local extremum, and thus v increases or decreases monotonically. Suppose $A(x)$ is such that it attains a minimum at $x = \bar{x}$. Then by (6) either

$$\frac{dv}{dx} = 0 \text{ or } v = c \text{ at } x = \bar{x}. \tag{10.6-7}$$

If the flow is subsonic for $x < \bar{x}$ and the velocity v has not attained the value c by $x = \bar{x}$, then it remains subsonic throughout. However, if at $x = \bar{x}$, $v = c$, then there are two possibilities: the flow becomes subsonic again for $x > \bar{x}$ or the flow becomes supersonic for $x < \bar{x}$.

It can be shown that continuous solutions for v and ρ are not possible for all values of the mass flux K_1. Solutions that are discontinuous at $x = \bar{x}$ correspond to shocks at $x = \bar{x}$. Let us consider a portion of the tube $a \le x \le b$. Suppose $p_a = p(a)$ and $p_b = p(b)$ denote the pressures at a and b. For a barotropic fluid, since the flow is steady and irrotational, it follows from (10.1-4) and (8.6-4), that

$$v(x)=\sqrt{2}\left\{\left[\frac{1}{2}v^2(a)+\frac{K\gamma}{\gamma-1}(\rho^{\gamma-1}(a)-\rho^{\gamma-1})+(\varpi(a)-\varpi(x))\right]\right\}^{\frac{1}{2}}$$
$$=\sqrt{2}\left\{\frac{1}{2}v^2(a)+\frac{K\gamma}{\gamma-1}\left[\left(\frac{Pa}{K}\right)^{\frac{\gamma-1}{\gamma}}-\left(\frac{p(x)}{K}\right)^{\frac{\gamma-1}{\gamma}}\right]+(\varpi(a)-\varpi(x))\right\}^{\frac{1}{2}}. \tag{10.6-8}$$

It follows from (2) and (8) that the mass flux K_1 is fixed by the pressure p and p_b. Equation (9), with the mass flux K_1 being given, can be viewed as an equation for the cross-sectional area A in terms of p, i.e., $A = \tilde{A}(p)$. It is easy to show that the function $\tilde{A}(p)$ has a minimum. However, if the cross-sectional area of the given tube is such that its minimum is less than that achieved by $\tilde{A}(p)$, then continuous solutions are not possible and we have a shock.[7]

10.7 One Dimensional Flows

We shall consider flows of a compressible fluid in Cartesian coordinates of the form

$$\dot{x}_1 = v(x_1, t), \quad \dot{x}_2 = 0, \quad \dot{x}_3 = 0, \tag{10.7-1}$$

with the density of the form

$$\rho = \rho(x_1, t). \tag{10.7-2}$$

Henceforth, we shall denote x_1 by x.

In the following analysis we shall neglect the effect of body-forces, but they can be included with little difficulty. On entering (2) into (1.4-5) we obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{10.7-3}$$

while substitution of (1) into (10.1-1) yields

$$\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial x}. \tag{10.7-4}$$

We shall investigate the possibility of traveling wave solutions to (3) and (4) and hence seek solutions of the form

$$v = v(x + ct) \text{ and } \rho = \rho(x + ct). \tag{10.7-5}$$

[7] A detailed treatment of shocks can be found in Von Mises and Friedrichs, *Fluid Dynamics*, chapter 5 (New York, Heidelberg, and Berlin: Springer-Verlag, 1971). R. Courant, and K.O. Friedrichs, *Supersonic Flow and Shock Waves*, (New York: Intersicence, 1948).

While the linearized equation (10.5-8) admits particular solutions that are functions of $x + ct$, here we are seeking such solutions to the system of nonlinear equations (3) and (4). We note that by virtue of (10.1-2), the pressure p is also a function of $x + ct$. If the relations (5) are invertible then v, ρ, and p can be expressed in terms of one another.

We shall find it convenient to rewrite (3) and (4) as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial \rho}\frac{\partial \rho}{\partial x} = 0 \tag{10.7-6}$$

and

$$\frac{\partial v}{\partial t} + \left(v + \frac{1}{\rho}\frac{dp}{dv}\right)\frac{\partial v}{\partial x} = 0. \tag{10.7-7}$$

Using the chain rule, (7) can be expressed as

$$\left[\frac{\partial \rho}{\partial t} + \left(v + \frac{1}{\rho}\frac{dp}{dv}\right)\frac{\partial \rho}{\partial x}\right]\frac{dv}{d\rho} = 0, \tag{10.7-8}$$

and if $\frac{dv}{d\rho} \neq 0$, then we immediately obtain

$$\frac{\partial \rho}{\partial t} + \left(v + \frac{1}{\rho}\frac{dp}{dv}\right)\frac{\partial \rho}{\partial x} = 0. \tag{10.7-9}$$

On comparing (6) and (9), we obtain

$$\left(\frac{dv}{d\rho}\right)^2 = \frac{c^2}{\rho^2}. \tag{10.7-10}$$

Integration yields

$$v = \pm \int_V \frac{c}{\rho} d\rho dv = \pm \int_V \frac{dp}{\rho c} dv. \tag{10.7-11}$$

Equation (11) gives a relationship between v, ρ, and p.

Invertibility of (5) allows us to write

$$x = \hat{x}(v, t), \tag{10.7-12}$$

and it follows from (7), (10) and (12) that

$$\frac{\partial \hat{x}}{\partial t} = v + \frac{1}{\rho}\frac{dp}{dv} = v \pm c(v), \tag{10.7-13}$$

which can be integrated to yield

$$\hat{x} = t[v \pm c(v)] + g(v), \tag{10.7-14}$$

where g is some arbitrary function of v. Equation (14) implies that the point at which the velocity is some fixed value moves with a constant velocity.

10.8 Equilibrium of a Gaseous Mass under Self-gravitation

We shall investigate whether a compressible barotropic fluid under the action of a self-gravitational body-force field can maintain itself in static equilibrium. Such a problem has relevance to the equilibrium states of gaseous stellar structures.

As we seek static solutions in a self-gravitating field, *Euler's dynamical equation* (9.1-1) reduces to

$$-\operatorname{grad} p - \rho \operatorname{grad} \varpi = 0, \tag{10.8-1}$$

with

$$\Delta \varpi = 4\pi\rho\, g, \tag{10.8-2}$$

where ϖ is the potential for the self-gravitational body-force field and g is the gravitational constant.

We shall find it convenient to study this problem within the context of a spherical coordinate system. We first notice that by virtue of (9.1-2), equations (1) and (2) can be effectively replaced by an equation in ϖ. On assuming a radially symmetric solution of the form

$$\varpi := \varpi(r), \tag{10.8-3}$$

by virtue of (1) and (2) we obtain

$$\frac{d^2\varpi}{dr^2} + \frac{2}{r}\frac{d\varpi}{dr} + \varpi^{\left(\frac{1}{\gamma-1}\right)} = 0. \tag{10.8-4}$$

We shall seek a solution that assumes that the static state of the gaseous mass is a sphere of radius R, the radius of which is not known *a priori*, and outside the sphere there is no gas, *i.e.*,

$$\rho = 0, \quad r > R. \tag{10.8-5}$$

The potential ϖ can be appropriately normalized so that

$$\varpi(R) = 1. \tag{10.8-6}$$

Also, as there is no gas outside the sphere, we shall assume that

$$\frac{d\varpi(R)}{dr} = 0. \tag{10.8-7}$$

Equation (4) has to be solved subject to the boundary conditions (6) and (7).

If the barotropic fluid is such that $\frac{6}{5} < \gamma < 2$, then it can be shown that

$$\varpi(r) = C(R-r)^{\frac{1}{\gamma-1}}\left[1 + P\left(R-r, (R-r)^{\left(\frac{\gamma}{\gamma-1}\right)}\right)\right], \tag{10.8-8}$$

where C is a constant and P is a double series with a finite radius of convergence. It immediately follows from (2) and (8) that

$$\rho \mathrel{\overset{\sim}{=}} (\text{const})(R - r)^{\frac{1}{\gamma-1}}. \tag{10.8-9}$$

Thus we see that if $\frac{6}{5} < \gamma < 2$, then the density is a continuously differentiable function. However, when $\gamma = 1$, we do not have a solution of the form (6) as we can see that the expression blows up.

10.9 Equilibrium of a Gaseous Mass under a Constant Body-Force Field

Let us consider a one-dimensional body of a compressible barotropic fluid occupying the domain $]0, d[$ along the x-axis, under the action of a constant specific body-force field $\mathbf{b} = g\mathbf{i}$. Then it can be shown that[8] the state of rest is characterized by

$$\rho(x) = \left[C + \frac{g(\gamma - 1)}{k\gamma}x\right]^{\left(\frac{1}{\gamma-1}\right)}, \tag{10.9-1}$$

where the constant C is determined by knowing the mass m of the gaseous body, *i.e.*,

$$\int_0^d \rho(x)dx = m. \tag{10.9-2}$$

It follows from (1) that if $1 < \gamma < 2$, then it is possible that for sufficiently large values of g the density could vanish or even become negative for some $x \in]0, d[$. However, this depends on the value of the index γ and the value of the specific body-force. In an interesting treatment of this problem,[9] the regions of zero and negative densities have been associated with regions of cavitation, the density being positive elsewhere.

[8] H. Beirão da Veiga, "An L^p theory for n-dimensional stationary, compressible Navier-Stokes equations and the incompressible limit for compressible fluids: The equilibrium solution," *Computational Mathematics and Mathematical Physics*, 109 (1987): 229–248.

[9] V. Lovicar, I. Straskraba, Remark on cavitation solutions of stationary compressible Navier-Stokes equations in one dimension, *Czech Math. J.* 41 (1991): 653–92.

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[1.] Courant, R. and Friedrichs, K.O. *Supersonic Flow and Shock Waves*. New York: Interscience, 1948.

11

Singular Surfaces and Waves

11.1 Introductory Remarks

Thus far we have considered "smooth" fields. When the hypothesis of smoothness is relaxed, any sort of singularity is possible. When we confine our attention to potential theory of classical hydrodynamics, the singularities of interest are point sources, dipoles, vortex lines, double layers, and the like. In gas dynamics, we are interested in another kind of singularity, the singular surface. We shall now study the kinematical properties of singular surfaces.

We shall assume that these surfaces have a continuously turning tangent plane. Let $\mathcal{C}_\delta$ be a part of the boundary of a region, which we shall denote by $\mathcal{R}_+$, and let $\mathbf{x}$ be a point on $\mathcal{C}_\delta$. Let Ψ denote a scalar, vector, or tensor field. The field Ψ is said to be smooth in $\mathcal{R}_+$ if it is continuously differentiable in $\mathcal{R}_+$, if for every point $\mathbf{x}$ on $\mathcal{C}_\delta$ the fields $\Psi(\mathbf{y})$ and $\partial_\mathbf{y}\Psi(\mathbf{y})$ approach the limits $\Psi^+(\mathbf{x})$ and $\partial_\mathbf{x}\Psi^+(\mathbf{x})$ as $\mathbf{y} \to \mathbf{x}$ and if $\Psi^+(\mathbf{x})$ is differentiable on any path $\mathcal{P}$ lying on $\mathcal{C}_\delta$. Hadamard's lemma asserts that for a smooth field Ψ, the theorem of the total differential holds for the limit functions Ψ^+ and $\partial_\mathbf{x}\Psi^+$. That is, if the path $\mathcal{P}$ is described by the parametric equation $\mathbf{x} = \mathbf{x}(\ell)$, then

$$\Psi^+(\ell) = [\partial_\mathbf{x}\Psi^+(\mathbf{x})][\mathbf{x}'(\ell)], \tag{11.1-1}$$

where the prime denotes the derivative with respect to the argument. The term on the right-hand side of (1) takes on the appropriate form based on whether Ψ is a scalar, vector or tensor. For instance, if Ψ is a scalar, the operation on the right-hand side is a scalar product.

EXERCISE 11.1.1

Show that Hadamard's lemma holds for a scalar field Ψ, by letting $\mathcal{C}_\delta$ denote an orientable surface that is a common boundary separating two regions $\mathcal{R}_+$ and $\mathcal{R}_-$

in each of which Ψ is smooth. Thus, at a point $\mathbf{x}$ on $\mathcal{C}_\delta$, the limits Ψ^+ and Ψ^-, and likewise $\partial_\mathbf{x}\Psi^+$ and $\partial_\mathbf{x}\Psi^-$ exist but need not be equal. The jumps of Ψ and $\partial_\mathbf{x}\Psi$ are defined as the differences of these values, that is

$$[\Psi](\mathbf{x}) := [\Psi] := \Psi^+ - \Psi, [\partial_\mathbf{x}\Psi] := \partial_\mathbf{x}\Psi^+ - \partial_\mathbf{x}\Psi^-. \tag{11.1-2}$$

If both of these jumps are not zero, $\mathcal{C}_\delta$ is said to be *singular with respect to* Ψ. It is important to recognize that Ψ is required to be smooth on each side of the surface. The jumps possible across a singular surface are restricted in kind. Since $[\Psi]$ is a differentiable function of $\mathbf{x}$ on $\mathcal{C}_\delta$, applying Hadamard's lemma to Ψ^+ and Ψ^- and subtracting yields

$$[\Psi]' = [\partial_\mathbf{x}\Psi] \cdot \mathbf{x}'(\ell) \tag{11.1-3}$$

This is Hadamard's *condition of compatibility*, which relates the jump in Ψ to that in $\partial_\mathbf{x}\Psi$. Equation (3) states that the jump of the tangential derivative is the tangential derivative of the jump. The jump of the normal derivative is unrestricted.

An important consequence of (3) follows where Ψ is continuous. Then $[\Psi] = 0$ and (3) delivers

$$[\partial_\mathbf{x}\Psi] \cdot \mathbf{x}'(\ell) = 0 \tag{11.1-4}$$

for all paths on $\mathcal{C}_\delta$. Since $\mathbf{x}'(\ell)$ may be any vector tangent to $\mathcal{C}_\delta$, (3) requires that there is a quantity $\mathrm{a}(\mathbf{x})$ such that

$$[\partial_\mathbf{x}\Psi] = \mathrm{a}\mathbf{n}, \quad \mathrm{a} = [\operatorname{grad}\Psi \cdot \mathbf{n}], \tag{11.1-5}$$

where $\mathbf{n}$ is a vector normal to $\mathcal{C}_\delta$. This result is known as Maxwell's theorem, and it asserts that the jump of the gradient of a continuous field is normal to the singular surface. We shall adopt the sign convention that $\mathbf{n}$ denotes a unit normal vector pointing from $\mathcal{R}_-$ towards $\mathcal{R}_+$. The quantity a is called the amplitude of the discontinuity and is uniquely determined.

If Ψ is a vector field, Hadamard's condition of compatibility implies that

$$[\partial_\mathbf{x}\Psi][\mathbf{x}'(\ell)] = 0. \tag{11.1-6}$$

As before, since $\mathbf{x}'(\ell)$ can be any vector tangent to $\mathcal{C}_\delta$, equation (6) implies that there is a vector $\mathbf{a}$ such that the tensor $[\partial_\mathbf{x}\Psi]$ can be expressed as

$$[\partial_\mathbf{x}\Psi] = \mathbf{a} \otimes \mathbf{n}. \tag{11.1-7}$$

If the vector $\mathbf{a}$ is parallel to $\mathbf{n}$, the singularity is called *longitudinal* and if $\mathbf{a}$ is perpendicular to $\mathbf{n}$ it is called *transversal*.

It follows from (7) that for a vector field Ψ

$$[\operatorname{div}\Psi] = \mathbf{a} \cdot \mathbf{n}, \tag{11.1-8}$$

and thus the jump is longitudinal, while

$$[\operatorname{curl}\Psi] = \mathbf{a} \wedge \mathbf{n}, \tag{11.1-9}$$

and thus the jump is transversal. The identities (8) and (9) constitute Weingarten's theorem.

Since any vector $\mathbf{a}$ can be expressed as $\mathbf{a} = \mathbf{n}(\mathbf{a} \cdot \mathbf{n}) - \mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{a})$, the jump in div Ψ is the normal component of the jump in the gradient of Ψ, and the jump in curl Ψ is the tangential component of the jump in the gradient of Ψ. It follows immediately that the jump of a continuous lamellar field is longitudinal and the jump of a solenoidal field is transversal.

Finally, the jump in an mth gradient of a scalar Ψ is given by

$$[\partial_x^m \Psi] = \mathbf{a} \otimes \overbrace{\mathbf{n} \otimes \ldots . \otimes \mathbf{n}}^{(m-1)\text{times}}. \tag{11.1-10}$$

11.2 Motion of Singular Surfaces

Let us consider the motion with respect to the reference configuration κ:

$$\mathbf{x} = \chi_\kappa(\mathbf{X}, t). \tag{1.2-3}$$

The surface C_δ in the reference configuration κ is said to be a *singular surface of* n*th order* if it is singular with respect to some n^{th} derivative of χ_κ but all lower-order derivatives are continuous in a region containing C_δ in its interior. The surface C_δ is allowed to move in the reference configuration. We shall consider only singular surfaces that persist for a finite interval of time. Thus they may be regarded as surfaces in a four-dimensional space whose points are pairs ($\mathbf{x}$,t). Alternatively, the surface can be expressed in the form

$$f(\mathbf{x}, t) = 0.$$

Then the speed of the displacement of the surface C_δ in a direction normal to itself[1] is given by

$$S_0 = \frac{-f'}{|\,\text{grad}\, f\,|}. \tag{2.6-16}$$

We next appeal to Hadamard's lemma to arrive at the kinematical equations of compatibility in the ($\mathbf{x}$,t) space. Let us consider a tangent path that is always normal to the current configuration of C_δ, and let $\frac{\delta}{\delta t}$ denote the derivative with respect to

[1] At this juncture, students will profit by reviewing section 2.6 of volume 1 of C. Truesdell, *A First Course in Rational Continuum Mechanics* (New York: Academic Press, 1991).

time along that path. The rate of change of a function φ with respect to an observer moving along with the surface C_δ is given by

$$\frac{\delta\varphi}{\delta t} = \frac{\partial\varphi}{\partial t} + ((\operatorname{grad}\varphi)\cdot\mathbf{n})S_{\mathbf{n}}. \tag{11.2-1}$$

Then, on applying Hadamard's lemma, we obtain the kinematical compatibility condition

$$\left[\frac{\delta\varphi}{\delta t}\right] = \left[\frac{\partial\varphi}{\partial t}\right] + [(\operatorname{grad}\varphi)\cdot\mathbf{n}]S_{\mathbf{n}}. \tag{11.2-2}$$

Therefore, if φ is continuous,

$$\left[\frac{\partial\varphi}{\partial t}\right] = -[(\operatorname{grad}\varphi)\cdot\mathbf{n}]S_{\mathbf{n}}, \tag{11.2-3}$$

and thus the jump in the spatial derivative determines the jump in the time derivative as a function of the speed of displacement of the surface, and vice versa.

Singular surfaces that are associated with the motion are of particular importance in continuum mechanics. By virtue of the invertibility of χ_κ, the surface $f(\mathbf{x}, t) = 0$ can be expressed as $\hat{f}(\mathbf{X}, t) = 0$. The first representation is called the *spatial surface*, the second, the *material diagram*. We are interested in two kinds of surfaces:

(i) material surfaces where $\hat{f}$ is independent of time,

(ii) propagating surfaces, called waves.

Let N denote the normal to the surface C_δ in the reference configuration. Then the speed of propagation S_N is defined through

$$S_N := \frac{-\frac{\partial\hat{f}}{\partial t}}{|\operatorname{grad}\hat{f}|} = \frac{-\hat{f}'}{|\operatorname{grad}\hat{f}|}. \tag{11.2-4}$$

The quantity S_N does not have a simple geometric significance. In fact, the value of S_N depends on which instant of time is assigned the value zero. If the present instant of time is assigned the value zero, then

$$S := S_N|_{t=0} = \frac{-f}{|\operatorname{grad} f|} \tag{11.2-5}$$

is called the *intrinsic speed of propagation*, and this is the speed of the normal advance of a wave front relative to particles instantaneously situated upon it.

EXERCISE 11.2.1

If a point $\mathbf{x}$ on a surface C_δ is moving with a velocity $\mathbf{v}$, and if the velocity of a particle instanteously at $\mathbf{x}$ is $\dot{\mathbf{x}}$, then show that

$$S^+ = (\mathbf{v} - \dot{\mathbf{x}}^+)\cdot\mathbf{n},$$

$$S = (\mathbf{v} - \dot{\mathbf{x}}) \cdot \mathbf{n},$$
$$S^- = (\mathbf{v} - \dot{\mathbf{x}}^-) \cdot \mathbf{n}. \tag{11.2-6}$$

If $S \neq 0$, the singular surface C_δ propagates through the material and hence is called a *wave*.

11.3 Classification of Singular Surfaces

A singular surface of order 0 is one in which the motion χ_κ is itself discontinuous. Such singular surfaces can be used to describe discontinuities such as fractures, tears, and welds in solids and the breakup or coalescence of droplets in fluids. However, practically nothing is known about singular surfaces of order 0.[2]

A singular surface of order 1 is called a *shock wave*. If the tangential component of the velocity is continuous but $[\dot{\mathbf{x}} \cdot \mathbf{n}] \neq 0$, it is called a *longitudinal* shock wave. If $[\dot{\mathbf{x}} \cdot \mathbf{n}] = 0$ but $[\dot{\mathbf{x}}] \neq 0$, the singular surface is called a *vortex sheet*. In general, a singular surface of order 1 is a surface at which $\dot{\mathbf{x}}$ is continuous but some first derivative of $\dot{\mathbf{x}}$ fails to be continuous.

Singular surfaces of order 0 and 1 are called *strong singularities*. Singular surfaces of order 2 are called *acceleration waves*. At such a surface, $\mathbf{x}$ and $\dot{\mathbf{x}}$ are continuous but some second derivative of $\mathbf{x}$ fails to be continuous. Singular surfaces of order 2 or greater are called *weak singularities*. A singular surface of infinite order, according to Hadamard, is one such that on each side of the surface, the relevant quantities are different analytic functions, yet all their derivatives are continuous across the surface. While they offer interesting possibilities, such singularities have not been studied.

For a singular surface of order 1, when we set $\Psi = \mathbf{x}$ in (11.1-7) and use material variables, we get

$$[\mathbf{F}] = \mathbf{s} \otimes \mathbf{N}, \mathbf{s} = [\mathbf{F}/\mathbf{N}] \tag{11.3-1}$$

and

$$[\dot{\mathbf{x}}] = -S_{\mathbf{N}}\mathbf{s}, \tag{11.3-2}$$

where $\mathbf{s}$ is the singularity vector of the discontinuity in $\dot{\mathbf{x}}$ that is parallel to the jump of the velocity. However, its magnitude varies with the choice of the initial state and thus does not provide a measure of the strength of the singularity. Denoting

[2] A more detailed description of singularities of order n can be found in C. Truesdell and R. Toupin, *The Classical Field Theories, Handbuch der Physik* 3 (Berlin, Göttingen, and Heidelberg: Springer-Verlag, 1960).

by $\mathbf{s}^+$ and $\mathbf{s}^-$ the singularity vectors associated with S^+ and S^- in the regions $\mathcal{R}_+$ and $\mathcal{R}_-$, from (16) we have

$$[\dot{\mathbf{x}}] = -S^+\mathbf{s}^+ = -S^-\mathbf{s}^-, \qquad (11.3\text{-}3)$$

and thus

$$[S\mathbf{s}] = 0. \qquad (11.3\text{-}4)$$

It follows from (11.2-6) that

$$S_{\mathbf{n}}[\mathbf{s}] = [(\dot{\mathbf{x}} \cdot \mathbf{n})\mathbf{s}], \qquad (11.3\text{-}5)$$

and thus in the case of a vortex sheet we obtain

$$S_{\mathbf{n}} = (\dot{\mathbf{x}} \cdot \mathbf{n}), \qquad (11.3\text{-}6)$$

in the case of a stationary shock,

$$[(\dot{\mathbf{x}} \cdot \mathbf{n})\mathbf{s}] = 0. \qquad (11.3\text{-}7)$$

EXERCISE 11.3.1
If $\mathbf{F}^+$ and $\mathbf{F}^-$ denote the deformation gradients in $\mathbf{R}_+$ and $\mathbf{R}_-$, show that

$$\frac{\det \mathbf{F}^+}{\det \mathbf{F}^-} = \frac{\dot{\mathbf{x}}^+ \cdot \mathbf{n} - S_{\mathbf{n}}}{\dot{\mathbf{x}}^- \cdot \mathbf{n} - S_{\mathbf{n}}} = \frac{S^+}{S^-}. \qquad (11.3\text{-}8)$$

Thus, the passage of a shock wave of order 1 causes an abrupt change in volume, the ratio being the intrinsic speeds of propagation.

By the conservation of mass (I.4-5), equation (24) can be expressed as

$$[\rho S] = 0, \text{ or } [\rho(\dot{\mathbf{x}} \cdot \mathbf{n} - S_{\mathbf{n}})] = 0, \qquad (11.3\text{-}9)$$

which is known as the Stokes-Christoffel condition. Thus in isochoric motions shock waves of order 1 are impossible, and the passage of a vortex sheet of order 1 leaves the volume unchanged.

11.4 Singular Surfaces of Order 2: Acceleration Waves

Application of Maxwell's theorem to a singular surface of order 2 yields a formally simple relation in the reference configuration.

EXERCISE 11.4.1
Show that the jumps in $\nabla\mathbf{F}$, $\dot{\mathbf{F}}$ and $\ddot{\mathbf{x}}$ are given by

$$\begin{aligned}[\nabla\mathbf{F}] &= \mathbf{a}\otimes(\mathbf{F}^T\mathbf{n})\otimes(\mathbf{F}^T\mathbf{n}),\\ [\dot{\mathbf{F}}] &= -S\mathbf{a}\otimes(\mathbf{F}^T\mathbf{n}),\\ [\ddot{\mathbf{x}}] &= S^2\mathbf{a},\end{aligned} \tag{11.4-1}$$

where $\mathbf{n}$ is the unit normal to the present configuration of C_δ. In the above expressions, $\mathbf{a}$ is called the *vector amplitude*, and S is the intrinsic speed of propagation. If $\mathbf{a}$ is parallel to the normal to the singular surface C_δ, we say that the acceleration wave is *longitudinal*, and when $\mathbf{a}$ is perpendicular to the normal we call the acceleration wave *transversal.*

Equation (11.3-1) is called Hugoniot's *geometrical condition of compatibility* and reflects the assumption that the discontinuity is spread out over a surface at the instant of question. The equations $(1)_{2,3}$ are called Hadamard's *kinematical conditions of compatibility* and reflect the assumption that the singular surface C_δ persists instantaneously.

EXERCISE 11.4.2
Show that the jump in the velocity gradient $\mathbf{G}$ satisfies

$$[\mathbf{G}] = -S\mathbf{a}\otimes\mathbf{n} \tag{11.4-2}$$

and hence derive the jump in the vorticity $\boldsymbol{\omega}$ as

$$[\boldsymbol{\omega}] = -S(\mathbf{a}\wedge\mathbf{n}) \tag{11.4-3}$$

and the jump in the div $\dot{\mathbf{x}}$ as

$$[\operatorname{div}\dot{\mathbf{x}}] = -S(\mathbf{a}\cdot\mathbf{n}). \tag{11.4-4}$$

It follows from (3) and (4) that a longitudinal acceleration wave carries a jump in the expansion but leaves the vorticity unchanged, while a transversal acceleration wave carries a jump in the vorticity but does not allow for an expansion.

Singular surfaces of higher order are discussed in Truesdell and Toupin, *The Classical Field Theories*, but not much is known about them.

General Reference

[1.] Truesdell, C. and Toupin, R.A. *The Classical Field Theories*, *Handbuch der Physik* 3. Berlin, Göttingen, and Heidelberg: Springer-Verlag, 1960.

A

Some Elementary Results from Real Analysis

In analogy with geometry in a finite-dimensional setting, we need the concepts of distance and convergence for functions defined on an interval $[a, b]$. In general, given an abstract set C with elements $\{x, y, \dots .\}$, we say that a function $\rho : C \times C \to R$ defines a *metric* on C if the following properties hold:

$$\begin{aligned} &\text{a. } \rho(x,x) = 0 \text{ for all } x \in C, \\ &\text{b. } \rho(x,y) = \rho(y,x) \text{ for all } x, y \in C, \\ &\text{c. } \rho(x,y) \le \rho(x,z) + \rho(y,z) \text{ for all } x, y, z \in C. \end{aligned} \tag{A.1}$$

If C is a vector space, then a function $\|\cdot\|: C \to R$ defines a *norm* on C if the following conditions are satisfied:

$$\begin{aligned} &\text{a. } \|x\| = 0 \Leftrightarrow x = 0, \\ &\text{b. } \|x - y\| = \|y - x\| \text{ for all } x, y \in C, \\ &\text{c. } \|x - y\| \le \|x - z\| + \|z - y\| \text{ for all } x, y, z \in C. \end{aligned} \tag{A.2}$$

It is easy to see that a set with a norm also has a metric defined by $\rho(x, y) = \|x - y\|$. The converse is not true.

Let C be a set equipped with a metric ρ. We say that a sequence x_n of elements of C *converges* to an element $x \in C$ if $\rho(x_n, x) \to 0$. The concept of convergence is now used to introduce the idea of continuity of a function T defined between two sets C_1 and C_2 equipped with metrics ρ_1 and ρ_2. We say that T is *continuous* if for every sequence $x_n \in C_1$ that converges to $x \in C_1$ the sequence $T(x_n)$ converges to $T(x)$ in C_2. Finally, we need the concept of completeness of a metric space. Let C be a metric space equipped with a metric ρ. A sequence x_n of C is called a *Cauchy sequence* if for every $\varepsilon > 0$ there is an N, a positive integer, such that

$\rho(x_n, x_m) < \varepsilon$ whenever $m, n > N$. We say that a pair (C, ρ) is *complete* if every Cauchy sequence converges to an element in C.

In our discussion concerning the Nekrasov integral equation, we deal with the vector space $C[a, b]$ of continuous functions defined on the interval $[a, b]$. The usual pointwise addition of two functions and the scalar multiplication of a real number by a function renders C[a,b] a vector space. Next, we define the *sup-norm* on C by

$$\|f\| = \sup_{a \leq x \leq b} |f(x)|. \tag{A.3}$$

It is a simple exercise to show that all of the properties listed in (2) hold for the sup-norm. More important, $C[a, b]$ is a complete metric space under this norm. Although this fact is not as easily verified as the previous assertion, its proof is a standard theorem of real analysis.[1]

The proof of the existence of a wave of greatest height requires two other concepts of convergences, known as *weak* and *strong* convergence. The vector space in which these types of convergences become relevant for our discussion is the space $L^2[a, b]$. To define this vector space, let $f \in C[a, b]$. Define a norm for f by

$$\|f\|_2 = \left(\int_a^b f(x)^2 dx \right)^{\frac{1}{2}}. \tag{A.4}$$

It is not difficult to show that $\| \cdot \|_2$ satisfies the definition (A.2) of a norm. The space $L^2[a, b]$ is defined to be the completion of $C[a, b]$ in this norm (2), in other words, a function $f \in L^2[a, b]$ if and only if there is a sequence of continuous functions f_n such that

$$\lim_{n \to \infty} \int_a^b [f_n(x) - f(x)]^2 dx = 0. \tag{A.5}$$

A sequence f_n of functions in $L^2[a, b]$ is said to converge to $f \in L^2[a, b]$ in the *strong topology* if (5) holds. A sequence $f_n \in L^2[a, b]$ is said to converge to f in the *weak topology* if

$$\lim_{n \to \infty} \int_a^b [f_n(x) - f(x)] g(x) dx = 0, \text{ for every } g \in L^2[a, b]. \tag{A.6}$$

The relationship between weak and strong convergence is rather simple. Any sequencethat converges strongly will converge weakly. This fact follows from an

[1] The proof can be found in A.E. Taylor, *Introduction to Functional Analysis* (New York: John Wiley and Sons, 1961).

application of the Cauchy-Schwartz inequality: Given any two functions $f, g \in L^2[a, b]$, the *Cauchy-Schwartz inequality* states that

$$\int_a^b f(x)g(x)dx \leq \|f\|_2\|g\|_2. \tag{A.7}$$

Then, if f_n converges strongly to f we have

$$\left|\int_a^b [f_n(x) - f(x)]g(x)dx\right| \leq \|f_n - f\|_2\|g\|_2 \to 0 \tag{A.8}$$

by the Cauchy-Schwartz inequality. A sequence that converges weakly, however, does not generally converge strongly. A simple example is the sequence $f_n(x) = \sin(n\pi x)$ which converges weakly to $f \equiv 0$ in $L^2[0, 1]$ (this result is the well-known Riemann-Lebesgue lemma) while $\|\sin(n\pi x)\|_2 = \frac{1}{2}$ so that f_n does not converge strongly to the zero function.

Weak convergence has played a crucial role as a tool in establishing the well-posedness of the mathematical models of continuum mechanics because of the following fact: A sequence of functions f_n that is bounded (*i.e.*, $\|f_n\|_2 \leq M$, where M is independent of n) has a subsequence that converges weakly. In other words, the analog of the Bolzano-Weierstrass theorem of finite dimensional analysis holds true if one uses the notion of weak convergence. In applications, as will be the case of the problem at hand, it is possible to come up with a sequence of approximate solutions to one's problem whose total energy, say, is bounded. The foregoing result enables one to extract a subsequence that converges weakly to a function f that will be a candidate for the solution of the problem.

The next phase of the analysis is to show that this candidate indeed satisfies all the requirements of being a solution. For linear problems this approach has had a fabulous success. For nonlinear problems, however, a major difficulty arises since a function of a weakly convergent sequence does not necessarily converge to the function of the limit. For instance, in the example of the sine functions, f_n converges weakly to zero while f_n^2 converges weakly to $\frac{1}{2}$ (because $\sin^2(n\pi x) = \frac{1}{2} - \frac{1}{2}\cos(2n\pi x)$ and the cosine terms converge weakly to zero as before) not zero. One then has to work considerably harder to establish the strong convergence of the sequence of approximations and thus pass to the limit. How this is done for the equation that arises from Stokes conjecture is outlined in Chapter 9.

Another analytical tool used in Chapter 9 has to do with the relationship between the Fourier series of a function $\theta \in L^2[-\pi, \pi]$ and its *conjugate*. Let θ have the Fourier series

$$a_0 + \sum_{k=1}^{\infty}(a_k \cos ks + b_k \sin ks), \tag{A.9}$$

with the usual formulae for the coefficients a_k and b_k as the inner products of θ with the appropriate base functions. The trigonometric series

$$\sum_{k=1}^{\infty}(a_k \sin ks - b_k \cos ks) \tag{A.10}$$

is called the *conjugate Fourier series* of θ and is denoted by $C\theta$. The following lemma shows that the map between a Fourier series and its conjugate Fourier series is well defined and continuous when one confines the domain of C to L^2.

LEMMA A.1[2]. *Let $\theta \in L^2$. Then $C\theta \in L^2$ and there is a constant A such that*

$$\int_{-\pi}^{\pi} |C\theta(s)|^2 ds \leq A \int_{\pi}^{\pi} |\theta(s)|^2 ds. \tag{A.11}$$

[2]The proof of this result can be found in chapter 7 of A. Zygmund, *Trigonometric Series*, vol 1. (Cambridge University Press, 1959).

Solutions to Exercises

Chapter 5

5.1.1: In a viscometric flow (4.2-15) reduces to $\mathbf{A}_2 = 2\kappa^2 \mathbf{N}^T\mathbf{N}$.

To complete the exercise, use (4.2-30).

5.1.2: Obviously (5.1-24) is an instance of (5.1-23). Calculate $\mathbf{f}(\kappa, \mathbf{QNQ}^T)$ using (5.1-5) and verify that (4.22-10) is satisfied.

5.3.1: With respect to a viscometric basis, $\mathbf{N}$ has the representation (4.21-39) and $\mathbf{A}_1$ is given by (4.2-15). Substitute the expression into (4.4-25) and use (5.1-4) to obtain (5.3-1) and (5.3-2).

5.4.1: With respect to the natural basis, the shearing flow is given by $v = v(x_1)\mathbf{e}_2$, and $\mathbf{N} = \mathbf{e}_2 \otimes \mathbf{e}_1$. Thus (5.1-5) leads to

$$\begin{aligned} \mathbf{S} &= \hat{\tau}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \hat{\sigma}_1(\mathbf{e}_1 \otimes \mathbf{e}_1) + \hat{\sigma}_2(\mathbf{e}_2 \otimes \mathbf{e}_2), \\ \hat{\tau}(x_1) &:= \tau(v'(x_1)), \hat{\sigma}_1 := \sigma_1(v'(x_1)), \hat{\sigma}_2 := \sigma_2(v'(x_1)). \end{aligned}$$

Hence

$$\operatorname{div} \mathbf{S} = \hat{\tau}'\mathbf{e}_2 + \hat{\sigma}_1'\mathbf{e}_1, \tag{A.12}$$

so the vorticity equation (5.8-10) is satisfied if and only if $\tau'' = 0$. Thus (5.4-5) holds. Using (*) and (5.4-1), we obtain

$$-ae_2 + \hat{\sigma}_1' e_1 = \rho \text{ grad } \varphi.$$

Integration yields

$$\rho\varphi = -ax_2 + \hat{\sigma}_1(x_1) + h(t),$$

which is the result of eliminating $k(x_1)$ from (5.4-4) and (5.4-6).

5.4.2: Formulate div **T** in physical coordinates. It follows that $\partial_\theta \phi$ and $\partial_z \phi$ are constants, and this in turn leads to (5.4-24).

5.4.3: (5.4-32) follows directly from (5.4-22), (5.4-24) and (5.4-30).

5.5.1: Following from integration (5.5-1) set in the form (1.7-4) subject to the initial condition (1.7-5).

5.5.2: (5.5-7) follows trivially from the assumption (5.5-6) and (5.5-2).

5.5.3: Use (1.7-4) to rewrite (5.5-8) as

$$\begin{aligned}\dot{\xi}_1 &= -\Omega \xi_2 - F(x_3)\sin \Omega t,\\ \dot{\xi}_2 &= \Omega \xi_1 + F(x_3)\cos \Omega t.\end{aligned}$$

To solve this system it is best to use complex variables. Multiply the second equation by $i = \sqrt{-1}$, and add the outcome of the first equation to obtain

$$(\xi_1 + i\xi_2)^{\bullet} - i\Omega(\xi_x + i\xi_2) = F(x_3)[\sin \Omega t - i \cos \Omega t]$$

The above is subject to the condition $(\xi_1 + i\xi_2)(t) = x_1 + ix_2$. Then

$$\begin{aligned}\xi_1 &= x_1 \cos \Omega s - x_2 \sin \Omega s - F(x_3)\sin \Omega \tau,\\ \xi_2 &= x_1 \sin \Omega s - x_2 \cos \Omega s + F(x_3)\cos \Omega \tau,\\ \xi_3 &= x_3,\end{aligned}$$

where $s := \tau - t$. Thus

$$[F_t(\tau)] = \left\| \begin{matrix} \cos \Omega s & -\sin \Omega s & \kappa s \sin \Omega \tau \\ \sin \Omega s & \cos \Omega s & \kappa s \cos \Omega \tau \\ 0 & 0 & 1 \end{matrix} \right\|,$$

where $\kappa := F'(x_3)$. For a rotation of angle $\Omega\tau$ about the x_3-axis, we find that

$$\mathbf{F}_o(\tau) = \mathbf{Q}(1 + \kappa \tau \mathbf{N}_o),$$

$\mathbf{N}_o$ being given by (4.2-36).

5.6.1: Substituting (4.2-35) into (1.3-3) we obtain (5.6-4).

5.6.2: Substitute for h_1 and h_2 in $(5.6\text{-}3)_1$ and $(5.6\text{-}3)_2$ from (5.6-6) and integrate the equations (5.4-1) by use of (5.6-3) and (5.6-4).

5.6.3: The condition that the fluid adheres to the plates implies that

$$\dot{x}_1 - \frac{1}{2}\Omega a = -\Omega x_2, \dot{x}_2 = \Omega x_1, \dot{x}_3 = 0 \text{ on the plane } x_3 = d,$$
$$\dot{x}_1 + \frac{1}{2}\Omega a = -\Omega x_2, \dot{x}_2 = \Omega x_1, \dot{x}_3 = 0 \text{ on the plane } x_3 = 0,$$

now using (4.2-35).

5.6.4: If (5.6-11) holds, then $f' = \frac{a}{d}$, $f'' = g' = g'' = 0$, so both sides of (5.6-9) with ϖ^2 neglected are constant.

5.7.1: Beginning with (1.7-4) and with the initial conditions (1.7-5) results in the expression for the transplacement.

5.7.2: Bearing in mind that μ, σ_1, σ_2, and ∇v are functions of $\mathbf{p}$ alone, use the identity

$$\operatorname{div}(\mathbf{u} \otimes \mathbf{v}) = [\operatorname{grad} \mathbf{u}]\mathbf{v} + (\operatorname{div} \mathbf{v})\mathbf{u}$$

to show that

$$\operatorname{div}(\mu(\mathbf{k} \otimes \nabla v + \nabla v \otimes \mathbf{k})] = \operatorname{div}(\mu \nabla v)\mathbf{k}, \tag{A.13}$$
$$\operatorname{div}[\sigma_2(\mathbf{k} \otimes \mathbf{k})] = \mathbf{0}, \tag{A.14}$$
$$\operatorname{div}\left[\frac{\sigma_1}{\kappa^2}(\nabla v \otimes \nabla v)\right] = \left[\operatorname{div}\left(\frac{\sigma_1}{\kappa^2}\nabla v\right)\right]\nabla v + \frac{\sigma_1}{\kappa^2}[\nabla\nabla v]\nabla v. \tag{A.15}$$

Then, since $\nabla\nabla v$ is symmetric,

$$[\nabla\nabla v]\nabla v = [\nabla\nabla v]^T \nabla v = \nabla\left(\frac{1}{2}\nabla v\right)^2 = \kappa\nabla\kappa.$$

These above relations obtain (5.7-5).

5.7.3: Use (5.7-4) to calculate $T_{(zz)}$, and then use (5.7-6) and (5.7-9) to obtain (5.7-16). (5.7-17) follows trivially from (5.7-16).

Chapter 6

6.1.1: Trivially follows from specializing (4.1-3) and recognizing the definition for A_n given in (2.2-12). The dependence on C_t^t is to be replaced by the dependence on the derivatives $\partial_\lambda^m C_t(\lambda)|_{\lambda=t}$, for $m = 1, 2, \ldots, n$.

6.1.2: In a rigid motion, $\mathbf{A}_n = \mathbf{0}$ for every n, so (6.1-5) requires that $\mathbf{f}(0,0,\ldots,0)$ commute with every orthogonal tensor. The fact that it is spherical follows from Exercise 4.1 in Chapter 4.

6.1.3: It suffices to consider the case of simple shearing (2.2-13). Then

$$[\mathbf{A}_1] = \begin{Vmatrix} 0 & \kappa & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0, \end{Vmatrix},$$

$$[\mathbf{A}_2] = \begin{Vmatrix} 2\kappa^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix},$$

and the array of scalars listed as (6.1-7) becomes 0, $2\kappa^2$, 0, $2\kappa^2$, $4\kappa^2$, $4\kappa^4$, $8\kappa^6$, 0, $2\kappa^4$, 0, $4\kappa^6$. These calculations are straightforward.

6.1.4: Bearing in mind (4.2-15), because $\kappa\mathbf{N} = (-\kappa)(-\mathbf{N})$ we may use either κ or $-\kappa$ as r in (6.1-1), whichever is positive. Use (4.2-15) also in (6.1-19) to get (6.1-21). Calculate **S** from (6.1-18) and compare the outcome with (5.1-24) to obtain (6.1-22).

6.1.5: Use of (2.11-32) and (2.11-39) yield (6.1-24). Then use (2.11-33), (2.11-8), (6.1-24), (2.11-37), (2.11-39), and (2.11-33) to get (6.1-24). (6.1-17) evaluates the traction; then simplify outcomes using (6.1-24).

6.4.1: Use (6.4-5) and the fact that $\mathbf{u}_1 = 0$ to calculate $\mathbf{A}_1^2$. Then (6.3-7), (6.4-5), (6.4-3), and $\mathbf{u}_1 = 0$ give the indicated expression for $\mathbf{A}_2$. Evaluate $\mathbf{A}_1$ using (6.4-3) to $O(a^3)$, and find **t** and π by comparing (6.3-10) and (6.1-18). Use (6.3-12) to arrive at (6.4-15), and conclude (6.4-16) and (6.4-17) in just the same manner as step 1.

6.4.2: Substitute (6.4-18)–(6.4-20) into (6.1-18), using (6.1-19), and comparing with (6.3-10) yields (6.4-21) and (6.4-22).

6.4.3: Substitution of (6.4-21) and (6.4-22) into (6.3-12) yields the result.

6.4.4: The desired results follow from (6.4-23), (6.3-4), and (6.3-7).

6.4.5: To derive (6.4-32) and (6.4-33), use essentially the same procedure as in the previous exercises.

6.6.1: Substitution trivially from (6.6-3) and (6.6-5).

6.6.2: Direct substitution of (6.6-10) into (6.6-2) verifies that the flow is universal.

6.6.3: Use (6.6-5), (6.6-20), (6.6-17), and (6.6-18) in (6.6-16) to verify that (6.6-19) is satisfied.

6.6.4: A simple rearrangement of the terms on the left-hand side of (6.6-21) shows that it can be expressed for plane flows as the gradient of $tr\mathbf{A}_1^2$.

6.6.5: The identities (6.6-33) follow from an elementary rearrangement of the terms on the left-hand side.

Chapter 7

7.2.1: Use of (5.5-1) and (6.1-17) in Cauchy's First Law (5.4-1) defines $\hat{p} := p - (2\alpha_1 + \alpha_2)\left(\frac{\partial u}{\partial y}\right)^2$ and $c(t) := \frac{\partial \hat{p}}{\partial x}$ to verify (7.2-1).

7.2.2: For v given by (7.2-2) compute $\frac{\partial v}{\partial x}$ and its maximum for a fixed value of V and ωt.

7.2.3: Find a^2 from (7.2-5), and use $\alpha_1 > 0$ while taking the limit $\xi \rightarrow 0$ to establish (7.2-8). When $\alpha_1 < 0$, the maximum for a^2 is at $\xi = -\frac{1}{\sqrt{3}}$, and its value is given by (7.2-9).Next use (7.2-6) to compute ω_{crit} corresponding to $\xi = -\frac{1}{\sqrt{3}}$ and thereby establish (7.2-10).

7.2.4: Use (6.1-17) and (3.6-2) to get (7.2-11). Substitute (7.2-2) into (7.2-11) to verify (7.2-12).

7.2.5: Substitute (7.2-13) into (7.2-1); verify that the identity is met if and only if (7.2-15) holds. Verify that (7.2-13) satisfies (7.2-14) by inspection.

7.2.6: Set $v = u_1(x) + u_2(x,t)$ and substitute this expression into (7.2-1) after setting $c \equiv 0$. Use separation of variables to express $u_2(x,t) = f(t)g(x)$. Finally, use the boundary condition (7.2-33) for $u_2(x,t)$. Equation (7.2-34) follows from a straightforward calculation.

7.3.1: Verify that (7.3-7) is a solution to (7.3-6) by direct substitution.

7.3.2: Substitute (7.3-8) into (7.3-6) and see that the equation holds if (7.3-9) is met.

7.3.3: Substitute (7.3-10) into (7.3-6) and observe that the equation is met if (7.3-11) holds.

7.3.4: Substitute (7.3-12) into (7.3-6) and observe that (7.3-13) provides a necessary condition for (7.3-6) to hold.

7.4.1: (7.4-1) follows from substituting (6.3-2) into (5.4-1) and using (7.3-2) and (7.3-3).

7.4.2: Substitute (7.4-14) into (7.4-6) and the outcome into (7.4-11). Set $k = -1$ in the resulting expression to (7.4-15) and (7.4-16).

7.4.3: Substitute (7.4-19) into (6.1-17), to obtain Cauchy's first law (5.4-1), and conclude that (7.4-24) has to hold.

7.4.4: Results (1)–(4) follow trivially from (7.4-26) and (7.4-27)

7.4.5: The result follows from (7.4-27), (7.4-20), and (7.4-29).

7.5.1: A complex function $F(x) := f(z) + ig(z)$, where $i = \sqrt{-1}$. Multiply $(7.5\text{-}2)_1$ by i and add $(7.5\text{-}2)_2$ to the outcome. Solve the resulting complex differential equation subject to $F(d) = 0$, $F(-d) = 0$ and $F(0) = 1$. Then (7.5-5) are the real and imaginary parts of $F(z)$, with m, n and δ as defined in the exercise.

7.5.2: Use the method outlined for the previous problem, subject to appropriate boundary conditions.

7.5.3: (7.5-7), (4.21-53), and (6.1-17), so (3.4-1) finds that the components of the traction vector are given by (7.5-19).

7.6.1: Extremize $tr\mathbf{A}_1^3$ subject to the constraint $tr\mathbf{A}_1 = 0$. A simple procedure would be to use the method of Lagrange multipliers.

7.6.2: The first of the equalities in (7.6-6) follows from the fact the v is solenoidal and the application of the divergence theorem to the term $tr[(\nabla v)(\nabla v)^T]$, which can be expressed as the $\text{div}[[\nabla v]v]$, and then using the fact that $v = 0$ on the boundary. The second equality follows in a like manner.

7.6.3: The Poincaré inequality (7.6-7) can be found in most standard books on partial differential equations. The difficulty rests in obtaining optimal Poincaré constants C_p.

7.6.4: (7.6-17) follows from (7.6-16), Holder's inequality, and the fact that the domain V is compact.

Chapter 8

8.1.1: (4.7-14) computes the stress power defined through (3.6-2). It immediately follows that $\mu > 0$ if and only if the stress power is nonnegative.

8.3.1: First show that the identity (2.11-9) can be expressed as

$$\ddot{x} = \dot{x}' + w \times \dot{x} + \text{grad}(\frac{1}{2}\dot{x} \cdot \dot{x}).$$

For an irrotational flow, in a simply connected domain $\dot{x} = -\text{grad } P_a$. Thus

$$\ddot{x} = \text{grad}\left(-P_1' + \frac{\dot{x}^2}{2}\right) = -\text{grad } P_a.$$

(8.3-3) immediately follows from (8.3-1).

8.3.2: If $\mathbf{W}'$ is zero, then $\dot{\mathbf{x}} = \text{grad } Q$ if the region is simply connected. (2.11-9) immediately implies that $\ddot{\mathbf{x}} = \text{grad}\left(Q + \frac{\dot{\mathbf{x}}^2}{2}\right) + 2W\dot{\mathbf{x}}$.

8.3.3: It follows from (8.3-4) that

$$\ddot{\mathbf{x}} = \text{grad}\left(Q + \frac{\dot{\mathbf{x}}^2}{2}\right) + 2W\dot{\mathbf{x}} = -\,\text{grad}\,\varphi + 2\nu\ \text{div}\,\mathbf{W};$$

the second equality holds in because of (8.2-1). Since the curve C parametrized as $\mathbf{x}(s)$ is normal to both $\dot{\mathbf{W}}\mathbf{x}$ and $\text{div}\,\mathbf{W}$, we obtain $\text{grad}\left(Q + \frac{\dot{\mathbf{x}}^2}{2} + \varphi\right) \cdot \frac{d\mathbf{x}(s)}{ds} = 0$, which implies that $\frac{d}{ds}\left(Q + \frac{\dot{\mathbf{x}}^2}{2} + \varphi\right) = 0$ and thence (8.3-5).

8.6.1: Substitute (8.6-2) into (8.1-1) to obtain

$$\frac{\partial p}{\partial x_2} = pff' - \mu f'',$$
$$\frac{\partial p}{\partial x_1} = \rho x_1[\upsilon f''' + ff'' - f'^2].$$

It immediately follows that

$$p = -\frac{1}{2}\rho[f^2 + A^2x_1^2] + \text{const.},$$

where A is as defined in (8.1-3).

8.6.2: It follows from Exercise 6.1 and the definition (8.6-5) that

$$\frac{\partial p}{\partial x_1} = \rho x_1 A^2[F''' + FF'' - F'^2].$$

Next, as $\eta \to \infty$, as $F' \to 1$ and $F''' + FF'' \to 0$, we find that $\frac{\partial p}{\partial x_1} \to -\rho x_1 A^2$, the pressure gradient that corresponds to the irrotational flow.

8.7.1: Substitute (8.7-5) into $rw = (r^2\omega)'$ and integrate.

8.9.1: It is straightforward to verify that (8.9-1) represents an isochoric flow.

8.9.2: Substituting (8.9-3) into (8.9-2) delivers an ordinary differential equation in G that can be easily written in the form of a hypergeometirc differential equation. (See E.L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.)

8.9.3: Plot the streamlines. Also use (8.9-4) to conclude that the streamlines experience a minimum when $\eta = 1/a$.

8.10.1: The result follows from the assumed form for the flow field (8.10-1), the fact that the axes of rotation are located at $(a, 0, d)$ and $(-a, 0, -d)$, and the condition of adherence.

8.12.1: Substitute (8.12-2) into (8.12-1) and the outcome into (8.1-1).

8.12.2: Equation (8.12-7) follows trivially from (8.12-3).

8.12.3: Equation $(8.12\text{-}10)_1$ follows from substituting (8.12-2) into $(8.12\text{-}1)_1$, making use of the definition of $\psi := rF(x)$, $R := r\sin\theta$ and $x := \cos\theta$. Similarly $(8.12\text{-}10)_2$ follows from $(8.12\text{-}1)_2$ and the definitions of ψ, R, and x.

8.12.4: That (8.12-11) and (8.12-12) are equivalent can be verified by direct calculation. The rest of the exercise follows straightforwardly.

8.12.5: Equation $(8.12\text{-}18)_2$ can be integrated to yield

$$\Omega' = A \exp\left(-2\int_0^x f\,dx\right),$$

where A is a constant. It immediately follows that Ω increases monotonically from 0 to C, as Ω' has the same sign.

8.13.1: Use integration by parts, the divergence theorem, the boundary condition (8.13-2), and the fact that $\operatorname{div}\boldsymbol{\delta}\dot{\mathbf{x}} = \mathbf{0}$ to show that

$$\int_v \delta\dot{x}\cdot D\delta\dot{x}\,dV = -\int_v \dot{\mathbf{x}}\cdot \mathbf{G}\boldsymbol{\delta}\dot{\mathbf{x}}\,dV.$$

Next, show that

$$\int_v \delta\dot{x}\cdot G^T\delta\dot{x}\,dV = 0$$

by appealing to integration by parts, the divergence theorem, and the fact that $\operatorname{div}\boldsymbol{\delta}\dot{x} = 0$. The result immediately follows from these identities.

8.13.3: Equation (8.13-33) follows trivially from integrating (8.13-25) twice.

8.16.1: The uniqueness follows from the fact that Navier's dynamical equations reduce to a linear equation subject to the boundary conditions (8.16-3).

8.16.2: (8.16-11) is a solution to the appropriate dynamical equations and the boundary conditions (8.16-10). Once again the uniqueness of the solution follows for the special subclass of flows of the form sought, as the equations are linear.

8.17.1: Use the statement by direct substitution.

8.17.2: Equation (8.17-2) is a linear partial differential equation, and its solution subject to (8.17-7) and (8.17-8) can be determined in a variety of ways and can be found in an introductory text in partial differential equations. Also see the *Handbuch* article by Berker for a discussion of solutions relevant to problems in fluid mechanics.

8.17.3: The foregoing comments apply here as well.

8.18.1: It follows from a straightforward application of separation of variables to (8.18-6) subject to the boundary condition (8.18-7) and the initial condition (8.18-8).

8.19.1: Substitute, (8.19-1) into (8.1-1) to find that f and g are governed by (8.6-3) and (8.19-2).

8.20.1: Use direct substitution.

8.21.1: Substitute (8.21-1) into (8.1-1), and take the curl of the outcome to obtain (8.21-2).

8.21.2: (8.21-5) meets (8.1-1) subject to the boundary conditions $(8.21\text{-}4)_{1,2}$.

Chapter 9

9.6.1: First show that $W_a = 0$ is both necessary and sufficient for Helmholtz's second vorticity theorem to hold. Next, Cauchy's vorticity formula implies that $W_a = 0$, and thus Helmholtz's second vorticity theorem follows from Cauchy's vorticity formula.

Next show that the D'Alembert-Euler condition (2.11-43) is a necessary and sufficient condition for Helmholtz's third vorticity theorem. However, the D'Alembert-Euler formula is a consequence of the Cauchy vorticity formula, and thus the third vorticity theorem of Helmholtz follows from the Cauchy vorticity formula.

Finally, Helmholtz's first theorem follows from Cauchy's vorticity formula in virtue of the latter implying the D'Alembert-Euler formula (2.11-43) (also see the solution to Exercise 13-7 of Chapter 2).

9.6.2: (9.6-1) and (9.6-2) follow from (2.11-45) and (2.11-47) and the chain rule, respectively. Show that (2.11-45) follows from (2.11-43) and thus (9.6-1) follows from (2.11-43).

9.7.1: It follows from $(9.7\text{-}1)_{1,2}$ that

$$\frac{\partial(x,z)}{\partial(\alpha,\beta)} = \det \begin{Vmatrix} 1+e^{-k\beta}\cos\kappa(\alpha+ct) & -e^{-k\beta}\sin\kappa(\alpha+ct) \\ -e^{-k\beta}\sin\kappa(\alpha+ct) & 1-e^{-k\beta}\cos\kappa(\alpha+ct) \end{Vmatrix}$$
$$= 1 - e^{-2\kappa\beta}.$$

9.7.2: (9.7-7) follows from (9.7-6) and the fact that $\kappa > 0$, $\beta > 0$, which imply that $e^{-\kappa\beta} < 1$.

9.7.3: A straightforward calculation using $(9.7\text{-}1)_{1,2}$ leads to

$$\dot{x} = \kappa c(z-\beta), \qquad \dot{z} = \kappa c(x-\alpha),$$

and the vorticity vector $\mathbf{w} := \text{curl } \dot{\mathbf{x}}$ is given by

$$\mathbf{w} = \mathbf{j}\{2\kappa c - \frac{\partial\beta}{\partial z} - \frac{\partial\alpha}{\partial x}\}.$$

It then follows from $(9.7\text{-}1)_{1,2}$ and (9.7-3) that

$$w = |\mathbf{w}| = \frac{2\kappa c e^{-2\kappa\beta}}{(1-e^{-2\kappa\beta})}.$$

Chapter 11

11: Let x_0 be any point on the surface S. We parametrize a sufficiently small neighborhood of x_0 through $\{A, \ell, \varpi\}$, where A is an ordered pair, ℓ is the parameter along the path P, and $\varpi = \varpi(x)$ is a smooth function such that $\varpi(x) = 0$ represents the equation for S near x_0. In particular, $\hat{\varpi}(x_0) = 0$. In terms of these parameters, we can express any y through $y = \hat{y}(A, \ell, \varpi)$, with $x_0 = \hat{y}(0,0,0)$ and $x(\ell) = \hat{y}(0, \ell, 0)$. Thus, $x'(\ell) = \frac{d\hat{y}}{dl}(0, \ell, 0)$. Now given any $\Psi = \hat{\Psi}(y)$, Hadamard's lemma follows from the chain rule for $\frac{\partial\Psi}{\partial\ell}$ by taking the limits $A \to 0$ and $\varpi \to 0$.

11.3.1: For a substantial surface given by $f(x,t) = 0$, we have by (1.8-2)

$$f' + (\text{grad } f)\cdot\mathbf{v} = 0,$$

where $\mathbf{v}$ is the velocity of the surface. Next, by the definition of S^+, we have

$$S^+ = \lim_{\substack{y\to x \\ y\in R^+, x\in s}} \frac{-\dot{f}}{|\text{grad } f|} = \lim_{\substack{y\to x \\ y\in R^+, x\in S}} \frac{[-f' - (\text{grad } f)\dot{x}]}{|\text{grad } f|}.$$

Since f, f', and grad f are continuous, we have

$$S^+ = \frac{-f'(x,t) - (\text{grad } f)\cdot \dot{\mathbf{x}}^+}{|\text{grad } f|} = (\mathbf{v} - \dot{\mathbf{x}}^+)\frac{\text{grad } f}{|\text{grad } f|} = (\mathbf{v} - \dot{\mathbf{x}}^+)\cdot \mathbf{n}.$$

The other two identities follow in a similar fashion.

11.3.1: It follows from (11.3-1) that

$$\det \mathbf{F}^+ = \det(\mathbf{F}^- + \mathbf{s}\otimes\mathbf{N}) = (\det \mathbf{F}^-)\det(\mathbf{1} + \mathbf{s}\otimes(((\mathbf{F}^-)^{-1})^T)\mathbf{N}).$$

Use the Cayley-Hamilton theorem and the preceeding expression to find that

$$\frac{\det \mathbf{F}^+}{\det \mathbf{F}^-} = 1 + \mathbf{s}\cdot(((\mathbf{F}^-)^{-1})^T)\mathbf{N}.$$

Use (2.6-21) to express this as

$$\frac{\det \mathbf{F}^+}{\det \mathbf{F}^-} = 1 + \mathbf{s}\cdot\mathbf{n}\frac{|\text{grad } f^-|}{|\text{Grad } f|}.$$

Use (11.2-6) and (11.3-2) in conjunction with the foregoing expression to obtain the result.

11.4.1: (a) Use (11.1-10) to obtain

$$[\text{Grad }\mathbf{F}] = \mathbf{a}\otimes\mathbf{N}\otimes\mathbf{N}.$$

Use (2.6-21) in this expression to find

$$[\text{Grad }\mathbf{F}] = \hat{\mathbf{a}}\otimes\mathbf{F}^T\mathbf{n}\otimes\mathbf{F}^T\mathbf{n},$$

where

$$\hat{\mathbf{a}} = \mathbf{a}\frac{|\text{grad } f|^2}{|\text{Grad } f|^2}$$

(b) Use (11.2.3), the result in (a) and (2.6-21) to get

$$[\dot{\mathbf{F}}] = -[\text{Grad } f]\mathbf{N}S_N = -S_N[\mathbf{a}\otimes\mathbf{F}^T\mathbf{n}]\frac{|\text{Grad } f|}{|\text{grad } f|}.$$

Next, use (2.6-25) to show that

$$[\dot{\mathbf{F}}] = -S\mathbf{a}\otimes\mathbf{F}^T\mathbf{n}.$$

(c) Use (11.2-3) to find

$$[\ddot{\mathbf{x}}] = [\text{Grad }\dot{\mathbf{x}}]\mathbf{N}S_N = [\dot{\mathbf{F}}]\mathbf{N}S_N.$$

Then use the result in (b), (2.6-21), and (2.6-25) to conclude that $[\ddot{\mathbf{x}}] = S^2\mathbf{a}$.

11.4.2: Follows in a manner identical to that used for exercise 4.1.

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